It disperses the fog to study the phenomenon of language in primitive kinds of applications in which one can command a clear view of the aim and functioning of the words. . . .

. . . but new types of language, new language-games as we may say, come into existence, and others become obsolete and get forgotten. . . .

What is the relation between thing and thing named?— Well, what *is* it? Look at [such and such] a language-game . . . or at another one: there you can see the sort of thing this relation consists in. . . .

Asking '. . . ?' *outside* a particular language-game is like what a boy once did, who had to say whether the verbs in certain sentences were in the active or passive voice, and who racked his brains over the question whether the verb 'to sleep' meant something active or passive. . . .

In order to see more clearly . . ., we must focus on the details of what goes on; we must look at them *from close to*. . . .

And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. . . .

. . . it is, rather, of the essence of our investigation that we do not seek to learn anything *new* by it. We want to *understand* something that is in plain view. For *this* is what we seem in some sense not to understand. . . .

A main source of our failure to understand is that we do not *command a clear view* of the use of our words. . . .

Our clear and simple language-games are not preparatory studies for a future regularization of language—as it were first approximations, ignoring friction and air-resistance. The language-games are rather set us as objects of comparison which

are meant to throw light on the facts of our language by way not only of similarities, but also of dissimilarities. . . .

For we can avoid ineptness or emptiness in our assertions only by presenting the model as what it is, as an object of comparison—as, so to speak, a measuring-rod; not as a preconceived idea to which reality *must* correspond. (The dogmatism into which we fall so easily in doing philosophy.)

The criteria which we accept . . . are much more complicated than might appear at first sight. That is, the games with [certain words], their employment in the linguistic intercourse that is carried on by their means, is more involved . . . than we are tempted to think. . . .

To understand a sentence means to understand a language. To understand a language means to be master of a technique. . . .

(Remember that we sometimes demand definitions for the sake not of their content, but of their form. Our requirement is an architectural one; the definition a kind of ornamental coping. . .). . . .

If language is to be a means of communication there must be agreement not only in definitions but also (queer as this may sound) in judgments. This seems to abolish logic but does not do so.—It is one thing to describe methods of measurement, and another to obtain and state results of measurement. . . .

((. . . James: "Our vocabulary is inadequate." Then why don't we introduce a new one? What would have to be the case for us to be able to?)). . .

It is as if one had altered the adjustment of a microscope. One did not see before what is now in focus. . . .

WITTGENSTEIN, *Philosophical Investigations*

INTENSION AND DECISION

A Philosophical Study

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—C—

For Frederic B. Fitch
this token of gratitude and esteem

PANDARUS . . . He that will have a cake out of the wheat, must needs tarry the grinding.

TROYLUS Have I not tarried?

PANDARUS Ay the grinding; but you must tarry the bolting.

TROYLUS Have I not tarried?

PANDARUS Ay the bolting; but you must tarry the leavening.

TROYLUS Still have I tarried.

PANDARUS Ay, to the leavening: but there's yet in the word hereafter, the kneading, the making of the cake, the heating of the oven, and the baking; nay, you must stay the cooling too, or you may chance to burn your lips.

Troylus and Cressida, I, 1

PREFACE

Logical syntax and semantics constitute a central part of modern philosophical theory. Syntax is concerned exclusively with the signs or expressions of a language and their interconnections. In semantics, on the other hand, or at least in one branch of it, we are interested not only in signs and their interconnections but also in the relations between signs and the objects which they designate or denote or stand for in one way or another. Such a semantics is called a *denotational* (or *designational* or *extensional*) semantics. A second branch of modern semantical theory is concerned not only with denotation but with the *meaning* or *intension* of expressions as well. A theory of meanings is sometimes called an *intensional* semantics. An intension of an expression is distinguished from its designatum or denotation or extension much as the connotation [connotatum] of a term is distinguished from its designatum in traditional logic. Frege's distinction between *Bedeutung* and *Sinn* is somewhat similar. As an example, the term 'man' may be said to denote severally individual men and to designate (or to have as its extension) the class of all men, whereas the connotatum of 'man' is neither an individual man nor the class of all men, but rather a new kind of entity altogether, a meaning or an intension.

We know a good deal about denotational semantics, thanks to the work of Carnap, Kotarbínski, Tarski, and others. In a sense, denotational semantics may now be regarded as a completed body of theory. The study of intensions, however, is in its infancy,

and although valuable progress has been made, no fully satisfactory semantical theory of intensions seems yet to have been formulated.

In traditional logic and in most modern theories, a term is regarded as having one and only one intension just as it is regarded as having one and only one extension or designatum. This traditional point of view has obscured the important fact that there are many different kinds of intensions to be discriminated carefully from one another. Traditional theories have failed to make such discrimination in part because they provide no clear condition under which two intensions differ or are the same. Each term (of the proper kind) has a unique intension, according to those theories, but precisely how this intension differs from the intension of some other term is not clearly indicated. The intension of 'man' is supposed to differ from the intension of 'animal', but just how we are not told. Also intensions are usually regarded as in some sense *sui generis*, and hence how they involve (or consist of or are generated out of) other kinds of entities is not considered.

It is not the deliberate aim of this book to discriminate as many kinds of intensions as possible. It will turn out, however, in the course of our inquiry that there are as a matter of fact many such kinds—with of course some family resemblance. In particular there is the family of *objective* intensions, which depend only or at least primarily upon logical and semantical features of the language at hand. These are objective in the sense that they in no way involve the *person* (who uses the language) or a *time*. As contrasted with these there is the family of *subjective* intensions of terms, which depend not only upon logical features of the language but also upon the particular user or users of the terms as well as upon the time or times at which they are used. Within each family there are many branches, and indeed we shall find ourselves somewhat surprised at the plethora of kinds of intensions we shall encounter.

Also we shall meet with entities which are very like intensions or which can be used to represent or mirror them for some purposes, but which are not in any proper sense intensions themselves. Such entities we shall call *quasi*-intensions. Here also there will be many types to distinguish, and the quasi-intensions will turn out to be every whit as important and interesting as intensions themselves.

The theory of objective intensions and quasi-intensions will be developed wholly in terms of the underlying denotational or designational semantics. As a result, intensions will not be regarded as entities *sui generis* but will find their rightful place among entities already

available. Intensional semantics is then seen to be not an independent branch of semantics but becomes wholly absorbed within the underlying theory of extension.

Of especial interest here are the subjective intensions and quasi-intensions, which have been almost wholly neglected in the recent semantical literature. These will be introduced not just casually as a result of common-sense reflection or "shallow" analysis, but upon the basis of the much recent experimental and mathematical work on utility and decision theory stemming from von Neumann and Morgenstern's *Theory of Games and Economic Behavior*, 2nd ed. (Princeton: Princeton University Press, 1947). The mutual adaptation and adjustment of the recent work on utility and decision to or with modern syntax and semantics results in a systematic theory of considerable interest and power.

A theory in which the user of a language is taken explicitly into account in one way or another is sometimes called a *pragmatics*. The general science of language, *semiotic*, then consists of syntax, semantics, and pragmatics together. In the author's *Toward a Systematic Pragmatics* (*Studies in Logic and the Foundations of Mathematics*, Amsterdam: North-Holland Publishing Co., 1959), the attempt was made to formulate a very narrow kind of theory. The pragmatics there was based primarily upon a relation of *acceptance* in the sense according to which one can say that a person X accepts or takes to be true a sentence a of a language L at a time t. This relation is merely classificatory. X either accepts a at t or not. We attempt in the present book to construct along somewhat similar lines a *comparative* and even *quantitative* pragmatics. We attempt in other words to define a locution 'X accepts at time t the sentence a of a language L *to degree* α', where α is some real or rational number between or including 0 and 1. In *Toward a Systematic Pragmatics* an outline of a theory of subjective intensions and quasi-intensions based on acceptance was given. But now in terms of acceptance to such and such a degree, a *quantitative* theory of subjective intensions and quasi-intensions may be developed of much greater theoretical power and flexibility. Here a notion of *preference* as between sentences as well as certain further notions of a more special sort are needed.

Chapter I is concerned with semantical preliminaries which are presupposed throughout, and there several kinds of objective quasi-intensions are introduced. Chapter II is devoted to the logic of preference, which likewise is needed in the later development. In Chapter III

a suitable adaptation of the fundamental notions of utility theory is presented in terms of which the notion of degree of acceptance is definable. On the basis of this the quantitative theory of subjective quasi-intensions is introduced in Chapter IV, together with some derivative notions of pragmatics and epistemology. In Chapter V the full theory of objective intensions and quasi-intensions is developed presupposing the simplified theory of types. Finally, in Chapter VI, a general synopsis of the whole theory is given, together with a few concluding comments.

More important here than the merely mathematico-logical detail is the over-all philosophic significance. Although logical method is employed fundamentally, this book would fail of its purpose if it were regarded merely as a treatise in logical theory. The purpose is to help explicate important notions of broad philosophic import. Some of these notions are semantical, but many belong to epistemology and the philosophy of science. Many of them also should prove to be of interest for ethics and the theory of value. It is hoped therefore that philosophers with genuine regard for the logical fundamentals of their subject will find some of the analyses given here helpful. Indeed, one of the abiding functions of logic is as a handmaiden to the philosopher. The sciences and working mathematics do well enough without it, but in philosophy logic is our one indispensable tool.

A maxim of much recent philosophical analysis is "Look not for the meaning; look for the use." In this book we attempt to develop specific techniques, coming directly out of recent decision theory, for the systematic study of use. But it is equally important to look for meanings—provided we know what they are. A precise ontology for meanings is given here for perhaps the first time. And indeed until we have such an ontology there would seem to be little point in looking for them. In view of this, that no precise ontology for meanings has been available, semantical foundations for philosophical analysis have seemed insecure to some, and semantics itself, in the sense of the theory of meaning or intension, has become somewhat suspect. But much of this suspicion seems to rest upon misunderstanding. When properly developed and suitably augmented by a pragmatics, the theory of intensions may be seen to provide the needed theoretical foundations for philosophical analysis.

We shall be concerned here primarily with semantical and pragmatical *theories* rather than with the application of these theories to specific linguistic situations. But it is now a commonplace to observe that with the help of abstract theories we best come to understand

concrete situations. This is not the occasion to defend the method of theoretical model-building in philosophy and the methodology of the sciences, but merely to note that this method is essential in semantics and pragmatics. Without an explicit theoretical model any approach to these subjects is liable to be haphazard, hit or miss, and perhaps even rather superficial or logically unsound.

Although the literature on intensions contains much that is valuable, most of the kinds of intensions we shall meet with here have not previously been studied upon a clear foundation in syntax, semantics, and decision theory. Nor have they been systematically interrelated with each other. It would be presumptuous therefore to suppose that this first tentative and exploratory attempt to formulate a systematic general theory of the many kinds of intensions and of their interrelations is wholly satisfactory or without flaw. To provide such a theory is a task for much cooperative work over many years of careful planning, planting, and pruning.

Of the authors whose work has been helpful in working out the theories here, mention should be made especially of Carnap and Leonard. Carnap's *Meaning and Necessity*, 2nd ed. (Chicago: University of Chicago Press, 1956) is a kind of *locus classicus* for semantical theories of intension, and the chapters on intension in Leonard's *Principles of Right Reason* (New York: Henry Holt and Co., 1957) are perhaps among the finest yet written. Also the author is indebted to the writings of Church, Fitch, Frege, Kneale, Nagel, Popper, Ryle, Quine, Russell, Tarski, and Woodger in various important ways. On utility theory mention should be made especially of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* and of the useful paper, "Outlines of a Formal Theory of Value, I" by D. Davidson, J. C. C. McKinsey, and P. Suppes in *Philosophy of Science* (22, 1955: 140–60).

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CONTENTS

I

SEMANTICAL PRELIMINARIES

In this chapter a brief account is given of the languages or language-systems which are to be under consideration throughout the first four chapters. Some of these systems are the *object-languages*, the precise logical structure of which is to be described. Part of this description is syntactical and part of it is semantical (in the sense of denotational semantics). The description of syntax is given within a *syntactical metalanguage*, that of semantics within a *semantical metalanguage* containing a syntactical metalanguage as a part. In the syntactical metalanguage we speak only of the signs or symbols of the object-language and of their various interrelationships. Within the semantical metalanguage, on the other hand, the signs of the object-language are interrelated with the objects with which the object-language deals. The primitive constants of the object-language are said to *designate* or *denote* certain objects and the variables are said to *range over* or to *take as values* the entities of certain well-knit domains of objects. Within a semantical metalanguage of this narrow kind there is no concern with meaning or intension *sui generis*, but only with designation, denotation, and allied notions.

The interesting fact is, however, that within a semantics of this narrow kind, as formulated for suitable object-languages, the

notion of *logical* or *analytic truth* may be defined. And on the basis of this notion, together with related ones, we may introduce by definition the so-called semantical *quasi-intensions* without further strengthening the underlying logic or semantical background, as we shall see.

The object-languages chosen, whose semantics is under discussion, are thought to be the most suitable kinds of logical systems for the purposes of philosophy and science. These systems are the so-called classical language-systems of *first order*. The underlying logic of such systems consists merely of the theory concerning the logical *connectives* 'or', 'not', 'and', 'if-then', and 'if and only if', together with the *quantifiers* and perhaps *identity*. No more complicated a logic is ordinarily needed. If a more complicated logic should be needed for some special purpose it can often be shown to be merely a special case of, or a further development of, this very simple kind of basic logic. Many of the *meta*languages here also will be of first order. A full syntactical and semantical description of the object-languages can be given within suitably formulated metalanguages of first order, so that for this purpose also no more complicated a logic is needed.

In §A and §B the first-order object-languages are outlined very briefly. In §C their syntax and denotational semantics are sketched. The concept of analytic truth is discussed in §D as well as some notions of Carnap's so-called *L-semantics*. In §E objective (or semantical) quasi-intensions are formally introduced. §F is devoted to the notion of a *quasi-proposition*. In §G some further types of objective quasi-intension are considered. For certain later purposes, some notions of *inductive logic* will be needed informally. These are described very briefly in §H.

The object-languages under discussion, their syntax, denotational semantics, and inductive logic, have been studied in much detail in the recent literature. Thus only a précis need be given here. For readers familiar with this material, §§A–D and §H of the present chapter are offered primarily for convenience of reference. Other readers will perhaps wish to supplement these sections with further details. §§E–G, on the other hand, are at least partially new.

In this first limning of the logic of intensions, most of the formulations to be met will be tentative and exploratory. Thus inevitably there will be some defects or other inadequacies of treatment here or there.

A. ONTOLOGY

Mathematicians who have concerned themselves with semantical matters have not, on the whole, it would seem, been interested in matters of ontology. Fundamental questions concerning what objects there actually are or are not somehow fail to attract them. The attention of the mathematician focusses primarily upon mathematical structure, and his intellectual delight arises (in part) from seeing that a given theory exhibits such and such a structure, from seeing how one structure is "modelled" in another, or in exhibiting some new structure and showing how it relates to previously studied ones, and so on. But in all of this the mathematician is satisfied so long as he has some "entities" or "objects" (or "sets" or "numbers" or "functions" or "spaces" or "points") to work with, and he does not inquire into their inner character or ontological status.

The philosophical logician, on the other hand, is more sensitive to matters of ontology and will be especially interested in the kind or kinds of entities there are actually. Likewise he will be concerned with the kind or kinds of entities needed for some given scientific, methodological, or philosophical purpose. He will not be satisfied with being told merely that such and such entities exhibit such and such a mathematical structure. He will wish to inquire more deeply into what these entities are, how they relate to other entities, how they are known or identified in experience, what role they play within knowledge generally, and so on. Also he will wish to ask whether the entity dealt with is *sui generis* or whether it is in some sense *reducible* to (or *constructible* in terms of) other, perhaps more fundamental, entities. Accordingly we may distinguish a *sui generis* and a constructivist approach to ontology. If one takes the *sui generis* view, one has, of course, the enormously difficult task of characterizing the new kind of entity *ab initio*. If one takes the reductionist or constructionist view, on the other hand, one's task may be no less formidable, but here one will explicitly show how the new kind of entity may be characterized in terms of entities already studied. The reductionist is free to use such entities as are already available and no more. Those who uphold the *sui generis* view are perhaps more liable to introduce a new kind of entity unnecessarily than to explore carefully interconnections between or among entities already at hand.

Suppose a biologist wishes to introduce the term 'entelechy', to take a notorious example, into the language of a certain area of biology. Suppose he introduces this as a primitive and finds that he needs also a new kind of entity with variables ranging over them. Also he gives axioms characterizing the new primitive and governing the new kind of variable. Of course these axioms should, to some extent at least, interrelate the new primitive and variables with the antecedent biological theory in a satisfactory and fruitful way. The reductionist, on the other hand, will introduce the term 'entelechy' if at all only on the basis of definitions and without new entities as values for variables. Entelechies (or whatever) would then emerge merely as special kinds of entities already available. The difficulties in developing satisfactorily and fruitfully either of these methods suggests that entelechies are not suitable objects for scientific study.[1]

Two important theories concerning the intension of terms have been put forward in recent years—the theories of Carnap and Church.[2] Both of these are built to some extent upon foundations suggested by Frege. Both regard or at least seem to regard intensions as *sui generis.* Both of them admit or seem to admit all kinds of new entities that are perhaps not actually needed. And both require some rather complicated primitive notions.

The philosophic logician sensitive to matters of ontology will wish to formulate a more economical theory, probably of a reductionist kind, in which such objects as meanings or intensions take their place among entities already available. The entities already available should be of the kind ordinarily admitted in the sciences and presupposed by our everyday speech and behavior. The kinds of entities admitted in most theories of intension, like entelechies, seem to have no legitimate place in science or scientific philosophy at all.

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[1] Cf., e.g., Carl G. Hempel, *Fundamentals of Concept Formation in Empirical Science* (*International Encyclopedia of Unified Science*, Vol. II, No. 7, Chicago: University of Chicago Press, 1952), p. 14 and *passim.*

[2] R. Carnap, *Meaning and Necessity*; and A. Church, "A Formulation of the Logic of Sense and Denotation," in *Structure, Method, and Meaning, Essays in Honor of Henry M. Sheffer* (New York: Liberal Arts Press, 1951), pp. 3–24, "The Need for Abstract Entities in Semantic Analysis," *Proceeding of the American Academy of Arts and Sciences* 80 (1951): 100–112, and *Introduction to Mathematical Logic*, Volume I (Princeton: Princeton University Press, 1956), esp. pp. 1–68.

It is not the task of the logical analyst to settle once and for all the fundamental problems of ontology. It is clearly his task, however, to stipulate as precisely as he can the ontology *required for specific purposes or analyses*. To stipulate an ontology is to describe or otherwise indicate the sort or sorts of objects or entities dealt with. In fact, following Quine in essentials, we may regard the ontology of a language or language-system as determined by the range or ranges of its variables.[3] Of course not all language-systems contain variables and for such systems this description is inappropriate.[4] But many systems do and it is only with such systems that we shall be concerned in this book.

Often a language or language-system deals with *many sorts* of entities. For each sort it is then frequently convenient to introduce a special kind of variable ranging over the entities of that sort.[5] This is especially desirable if the various sorts of entities differ remarkably from each other. The use of different kinds of variables helps to separate the different sorts of entities notationally as well as conceptually. And each kind of variable of course carries with it a separate kind of quantifier. Strictly it is not necessary to distinguish the various kinds of variables and quantifiers, and one is always free to reformulate a language containing many kinds of variables so as to contain a single kind. And conversely, a language with just one kind of variable can always be reformulated in such a way as to contain many kinds, if the requisite predicates are available within it.

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[3] Cf. W. V. Quine, *From a Logical Point of View* (Cambridge: Harvard University Press, 1953), *passim*, *Methods of Logic*, Revised edition (New York: Henry Holt and Co., 1959), *passim*, and *Word and Object* (Cambridge: The Technology Press of the Massachusetts Institute of Technology, and New York and London: John Wiley and Sons, Inc., 1960), *passim*. In discussions of existence we must be careful not to overlook the different roles played by bound and free variables. Some language-systems contain variables but without quantifiers upon them. For languages of this kind, Quine's dictum that to be is to be the value of a variable is not appropriate. In the systems under discussion here, quantifiers are introduced upon all kinds of variables admitted. Cf. the author's "On Denotation and Ontic Commitment," *Philosophical Studies* XIII (1962): 35–39.

[4] Cf. especially H. B. Curry and R. Feys, *Combinatory Logic*, Vol. I (*Studies in Logic and the Foundations of Mathematics*, Amsterdam: North-Holland Publishing Co., 1958).

[5] See especially A. Schmidt, "Über Deduktiven Theorien mit mehreren Sorten von Grunddingen," *Mathematische Annalen* 115 (1938): 485–506 and H. Wang, "Logic of Many-Sorted Theories," *The Journal of Symbolic Logic* 17 (1952): 105–116.

Many-sorted languages are especially convenient if one wishes to be as clear and economical as possible concerning the underlying ontology. The many sorts of objects dealt with are explicitly enumerated as the ranges of the different kinds of variables. One is not then tempted to include in the enumeration more sorts than are actually needed. On the other hand, if one collects all of one's entities together indiscriminately, one is less tempted to keep the different sorts separate, is more liable to gloss over important differences, and perhaps to admit more of them than are needed or actually exist. The logical analyst, with a robust sense of what actually is, will welcome the restraints which the use of many-sorted languages naturally imposes upon him.

In the course of our inquiry into the theory of intensions and quasi-intensions, we shall try always to be clear concerning the sorts of entities allowed as values for variables. In the first chapters these will always be entities of the following sorts: the *individuals* admitted in the object-language, the *expressions* of the object-language, the *users* (human or otherwise) of the language, *real numbers*, and *time-stretches*. In Chapter V, on the theory of types, *classes* of the individuals admitted in the object-language, *classes of these classes*, and so on, will also be recognized as entities.

It should be noted that this is in marked contrast to the usual procedure of regarding intensions as *sui generis*. Among the entities to be allowed here there is no sort singled out and labelled rather mysteriously 'Intensions'.[6] Intensions rather will emerge merely as special kinds of entities already provided for. This reductionist procedure has the advantage not only of admitting no new unnecessary entities for the theory of objective intensions and quasi-intensions, but also of dispensing with the need for any new primitive notions. In fact no new notions at all are needed for that theory and hence no new axioms or other assumptions, logical or nonlogical. For the theory of subjective intensions and quasi-intensions, some new pragmatical primitives are of course needed. But these are of a kind ordinarily admitted in the sciences and can be justified on their own merits, as we hope to show.

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[6] On mysterious labelling, cf. M. G. White, *Toward Reunion in Philosophy* (Cambridge: Harvard University Press, 1956), esp. p. 156ff.

B. MANY-SORTED LANGUAGE-SYSTEMS

The language-systems to be considered here as object-languages are first-order systems with (to simplify) just *one* kind of variable. The variables of this kind range over a certain sort or domain of individuals. The individuals may be selected more or less *ad libitum*, but once selected no further kinds of variables, ranging over classes of or relations between or among these individuals, are admitted. The very phrase 'first-order' is intended to suggest this kind of restriction. The *meta*languages to be formulated are also of first order but with *many* kinds of variables, each kind ranging over a distinct sort of entity. These many-sorted metalanguages are thought to be especially appropriate for semantical and pragmatical purposes.

In Chapter V an object-language based upon Russell's *simplified theory of types* will be considered. The metalanguage there also will be based upon type theory. Languages based upon type theory are no longer of first order but rather of infinite order. They may be regarded as languages with an infinite number of kinds of variables. They admit not only variables over a fundamental sort or domain of individuals, but also over further sorts, namely, over all classes of such individuals, all classes of such classes, and so on, and perhaps also over various kinds of relations.

Let us use '$\sim$' (read 'not') for *negation*, 'v' ('or') for *disjunction*, '$\supset$' ('if . . . then') for the *material conditional*, '.' ('and') for *conjunction*, and '$\equiv$' ('if and only if') for *material equivalence*, as is frequent in philosophical writings, all of these taken in the sense of classical, two-valued logic. '$\sim$' and 'v' we may regard as primitive, the others then being appropriately definable. In particular, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are any formulae, then

'$(\boldsymbol{A} \supset \boldsymbol{B})$' may abbreviate '$(\sim\boldsymbol{A} \vee \boldsymbol{B})$',

'$(\boldsymbol{A} \,.\, \boldsymbol{B})$' may abbreviate '$\sim(\sim\boldsymbol{A} \vee \sim\boldsymbol{B})$',

and '$(\boldsymbol{A} \equiv \boldsymbol{B})$' may abbreviate '$((\boldsymbol{A} \supset \boldsymbol{B}) \,.\, (\boldsymbol{B} \supset \boldsymbol{A}))$'.[7]

[7] Bold-face letters will occasionally be used informally, as here, as syntactical variables for expressions. Single quotation marks are then used ambiguously as ordinary quotes or as *quasi-quotes* in Quine's sense. Cf. his *Mathematical Logic*, Revised edition (Cambridge: Harvard University Press, 1951), pp. 33–37.

The quantifiers are symbolized with the help of parentheses. '(x)' (read 'for all x'), where 'x' is a variable, is a universal quantifier, and similarly for any other variable. The existential quantifier '$(\mathrm{E}x)$' ('there exists at least one x such that') may then be defined as '$\sim(x)\sim$', and similarly for any other variable. Quantifiers are allowed over all the kinds of variables admitted. The identity sign '$=$' ('is identical with') will also be included as a primitive logical sign. Axioms and rules concerning these logical notions, including *Modus Ponens* (*MP*) and *Generalization* (*Gen*), are needed to provide the usual principles of the classical, two-valued logic of truth-functions, quantifiers, and identity.[8]

For the moment, let us consider object-languages where all the variables are of just *one* kind. Let these variables be 'x', 'y', 'z', 'w', 'x'', 'y'', etc. These are the so-called *individual variables*, ranging over the fixed sort or domain of individuals. The nonlogical primitive *predicate constants*, each of specified *degree*, of the object-language or languages, are to be symbolized by such letters as 'P', 'Q', and 'R'. Also the systems may contain some primitive *individual constants*, each standing for or designating a distinct individual. We may let the nonitalic letters 'a', 'b', etc., be any such constants. Suppose 'P' and 'Q' are primitive one-place predicate constants. '$\mathrm{P}x$', 'Pa', '$\mathrm{Q}y$', '$x = \mathrm{b}$', etc., are then to be *atomic formulae*. More generally, any primitive predicate constant of degree n followed by n terms (i.e., primitive individual constants or variables) is an atomic formula. Also the result of writing a term followed by the identity sign '$=$' followed by a term, is an atomic formula. The *formulae* in general are then built up in the customary way from atomic formulae by means of negation, disjunction, and quantification. More specifically, if $\boldsymbol{A}$ is a formula then '$\sim\boldsymbol{A}$' is also. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are formulae, '$(\boldsymbol{A} \vee \boldsymbol{B})$' is. And if $\boldsymbol{A}$ is a formula and $\boldsymbol{x}$ a variable of the one kind, '$(\boldsymbol{x})\boldsymbol{A}$' is then also a formula.

We note that any occurrence of a variable $\boldsymbol{x}$ in a formula $\boldsymbol{A}$ in a context of the form '$(\boldsymbol{x})\boldsymbol{B}$', where $\boldsymbol{B}$ is a formula, is a *bound* occurrence of $\boldsymbol{x}$ in $\boldsymbol{A}$. An occurrence of a variable is then *free* if it is not bound. A *sentence* is a formula containing no free variables.

[8] For a fuller discussion of essentially this logic, in the appropriate kind of framework, see the author's *Truth and Denotation, A Study in Semantical Theory* (Chicago: University of Chicago Press, Toronto: University of Toronto Press, and London: Routledge and Kegan Paul, 1958), Chapter II.

Let us use Frege's assertion sign '$\vdash$' to stand for 'is a theorem'. '$\vdash A$', where A is a formula, is to read 'A is a theorem'.

The logical axioms for identity, already alluded to, are the following.

IdR1. $\vdash x = x$.

IdR2. $\vdash x = y \supset (A \supset B)$, where A is a formula containing (at least) one free occurrence of 'x' and B differs from A only in containing a free occurrence of 'y' in place of that free occurrence of 'x'.

Each primitive individual constant is to designate a unique individual, so that among the logical axioms we shall wish to include statements to the effect that these individuals are distinct from one another. Hence we have also that

IdR3. $\vdash \sim \mathrm{a} = \mathrm{b} \,.\, \sim \mathrm{b} = \mathrm{c} \,.\, \sim \mathrm{a} = \mathrm{c} \ldots\ldots$

Extensive use will be made of a notation for so-called *virtual classes*. Strictly there are no such entities. Certain kinds of one-place *abstracts*, however, may be introduced as providing a convenient notation and we may think of these abstracts as standing for a kind of virtual entity. The use of one-place abstracts will be convenient not only in the object-languages but in the metalanguages as well. To symbolize these an adaptation of Peano's inverted epsilon may be used. If $\boldsymbol{x}$ is a variable (of some kind) and A a formula,

$$\text{'}\boldsymbol{x} \ni A\text{'}$$

is to be a primitive one-place abstract. '$x \ni (\mathrm{P}x \vee \mathrm{Q}x)$' for example, where '$x$' is a specific individual variable and 'P' and 'Q' are primitive one-place predicate constants, stands for the virtual class of all individuals which have P or have Q. '$x \ni x = x$' stands for the virtual class of all self-identical individuals, i.e., all individuals.

Note that occurrences of a variable $\boldsymbol{x}$ in the context '$\boldsymbol{x} \ni A$', where A is a formula, are now to be regarded as *bound* occurrences, so that the foregoing definition of *bound occurrence* must be revised. Also the notion of *formula* must now be extended to allow primitively for occurrences of abstracts.[9]

Concerning abstracts we need certain *Rules of Abstraction* as additional logical rules. For the object-languages with just one kind of variable, we have the following rule.

[9] Cf. *Truth and Denotation*, p. 49 ff.

Abst. $\vdash y \ni \boldsymbol{A}\ \boldsymbol{x} \equiv \boldsymbol{B}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are any formulae and (i) $\boldsymbol{x}$ is an individual constant, $\boldsymbol{y}$ is a variable not free in $\boldsymbol{B}$, and $\boldsymbol{A}$ differs from $\boldsymbol{B}$ only in containing free occurrences of the variable $\boldsymbol{y}$ in one or more places where there are occurrences of $\boldsymbol{x}$ in $\boldsymbol{B}$, or (ii) $\boldsymbol{x}$ is an individual variable and $\boldsymbol{A}$ differs from $\boldsymbol{B}$ only in containing free occurrences of a variable $\boldsymbol{y}$ wherever and only where there are free occurrences of $\boldsymbol{x}$ in $\boldsymbol{B}$.

An analogous Rule of Abstraction is needed for each kind of variable introduced.

Two-place abstracts may be introduced similarly, with a corresponding *Rule of Two-Place Abstraction*. Such abstracts stand for virtual dyadic relations. And likewise for three-place abstracts, standing for virtual triadic relations, and so on.

Just as with the primitive predicate constants, we may speak of a one-place abstract as being of *degree one*, a two-place abstract as being of *degree two*, and so on.

Let '$x \ni (\mathrm{P}x \vee \mathrm{Q}x)$' be a one-place abstract as above. The formula '$x \ni (\mathrm{P}x \vee \mathrm{Q}x)\ \mathrm{a}$' then states that the individual a is a member of the virtual class of all individuals x such that Px or Qx. '$x \ni x = x\ \mathrm{a}$' states that a is member of the virtual class of all (self-identical) individuals. Note that one-place abstracts occur significantly with variables or individual constants as arguments just as primitive predicate constants do. Additional parentheses may be placed around the abstracts if desired. Thus we could write '$(x \ni (\mathrm{P}x \vee \mathrm{Q}x))\ \mathrm{a}$' in place of '$x \ni (\mathrm{P}x \vee \mathrm{Q}x)\ \mathrm{a}$' for additional clarity. But strictly these extra parentheses are not needed. And similarly for two-place abstracts.

Also it is convenient to take Russell's *singular descriptions*[10] as primitives of the object-languages. These are especially useful as a means for introducing additional individual constants by definition. We let

$$`(\iota \boldsymbol{x}.\boldsymbol{A})',$$

where $\boldsymbol{A}$ is a formula and $\boldsymbol{x}$ an individual variable, be significant

[10] See especially A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. I (Cambridge: Cambridge University Press, 1910 and 1925), pp. 172–86 and B. Russell, *Introduction to Mathematical Philosophy* (London: George Allen and Unwin, 1919), pp. 167–80.

primitively in the object-languages. '$(\iota x.A)$' may be read 'the one object x such that ----' where A is some given formula '----'.

If in fact there is one and only one object such that ----, where A or '----' is a formula containing 'x' as its *only* free variable, then '$(\iota x.A)$' stands for that object. But what do such expressions stand for if there is *no* such object or *more than one*? A convenient method for handling descriptions in these cases is provided by Carnap's adaptation of the method of Frege.[11] Carnap selects a certain individual a*, as the object described by descriptive phrases not satisfying the existence and uniqueness conditions. Let us also admit 'a*' then, following Carnap, as an additional primitive individual constant. Additional clauses must then be added to *IdR3* to the effect that $\sim a^* = a$, etc.

As a further rule of the object-languages we assume also the following *Rule of Descriptions.*

RDescr. $\vdash x \ni A\ (\iota y.B) \equiv ((Ex)(A\ .\ (y)(B \equiv y = x))\ v\ (\sim(Ex)(y)(B \equiv y = x)\ .\ x \ni A\ a^*))$, where x and y are distinct individual variables, A and B are any formulae, and there are no free occurrences of x in B.

It is often desirable to introduce abbreviations for expressions such as '$(\iota x.A)$', where A is a formula containing x as its only free variable if any. Such abbreviations, as well as the original expressions themselves, will hereafter be called *defined individual constants.* Strictly the ι-expressions are not constants, but complex phrases. But it will do no harm to refer to them as defined constants. Thus we might wish to introduce a new individual constant, 'Socrates' say, as an abbreviation for '$(\iota x.Hx)$', where 'H' is the predicate 'husband of Xantippe'. We should then speak of both 'Socrates' and '$(\iota x.Hx)$' as defined individual constants, even though in the latter case the usage is somewhat unusual.

Note that when ι-expressions are added as primitives, the notion of *formula*, as well as the preceding logical rules, must be suitably extended. Also we must note that occurrences of a variable x in contexts of the form '$(\iota x.B)$' are now to be regarded as *bound* occurrences. The ι-expressions are presumed handled as substituends (i.e., as substitutable)

[11] *Meaning and Necessity*, p. 37.

for variables. The logical principles presupposed must thus be worded in such a way as to allow this. Note that the left-hand side of the *Rule of Descriptions* contains '$(\imath y.B)$' only as an argument for a one-place abstract. But in view of the now extended *Abst*, and because descriptions are now substituends for variables, all possible contexts in which '$(\imath y.B)$' can significantly occur are in effect covered by this Rule.

The one-sorted, first-order language-systems with identity, virtual classes, and descriptions will be the object-languages throughout the first four chapters. In addition to the logical axioms and rules, these systems L also contain nonlogical axioms characterizing the specific subject-matter with which they deal.

All the semantical metalanguages to be considered contain essentially this same logic, but with *several* kinds of variables. The detailed description of these need not be given but will be clear enough as we proceed.

C. SYNTAX AND DENOTATIONAL SEMANTICS

The formal syntax of the language-systems L is to be handled on the basis of what is essentially Tarski's theory of *concatenation* as formulated within a first-order logic. We go on now to describe a *syntactical metalanguage* for L based on concatenation.

If a and b are any two expressions of L, $(a \frown b)$ is said to be the *concatenate* of them, i.e., the expression *a followed immediately* by b. Thus if a is '$(x)(\mathrm{P}x$' and b is '$\mathrm{v}\ \mathrm{Q}x)$', then the concatenate $(a \frown b)$ is '$(x)(\mathrm{P}x \ \mathrm{v}\ \mathrm{Q}x)$'.

Each primitive sign of L is to have a so-called *structural-descriptive* name as a primitive in the syntactical metalanguage. '$\sim$' for example is to have the structural description '*tilde*', 'v' '*vee*', and so on. Then the compound expression '$\sim$v' has the structural-descriptive name '(*tilde* $\frown$ *vee*)'. The use of concatenation and structural descriptions will help us to achieve exactitude and to avoid the sometimes confusing and ambiguous use of quotation marks within the syntactical metalanguage.

We let the *italic* letters '*a*', '*b*', '*c*', etc., possibly with primes or numerical subscripts, be the *only* variables of the syntax language and

we let these take the *expressions* of L, i.e., all primitive signs and all concatenates, as values. These variables will play in the formalized syntax and semantics of L essentially the role which the bold-face letters did in the informal syntax and semantics of the preceding section.

The syntax language must itself contain a logic, in fact, just the logic of the preceding section with the expressional variables as the one kind of variables. We thus have identity between expressions as a syntactical primitive as well as one-place abstracts with respect to the expressional variables. Such abstracts are called *expressional* abstracts.

Within the syntax language suitable rules of logic are presupposed, including principles of identity. Also specifically syntactical axioms and rules are presumed given, on the basis of which the appropriate syntactical properties of L, or rather of the expressions of L, may be provided. These are roughly as follows. First we assume that no primitive sign of L is the same as any other, so that

$$\sim vee = tilde, \text{ etc.}$$

No primitive sign of L is any concatenate, so that

$$\sim(\mathrm{E}a)(\mathrm{E}b)vee = (a \frown b), \quad \sim(\mathrm{E}a)(\mathrm{E}b)tilde = (a \frown b), \quad \text{etc.}$$

We must also specify the conditions under which two concatenates are identical. For this we assume that

$$\vdash (a \frown b) = (c \frown d) \equiv ((a = c \,.\, b = d) \vee (\mathrm{E}e)(b = (e \frown d) \,.\, c = (a \frown e)) \vee (\mathrm{E}e)(a = (c \frown e) \,.\, d = (e \frown b))).$$

Finally, we assume a Syntactical Rule of Infinite Induction, to the effect that if it is provable that a given property holds of *vee*, provable that it holds of *tilde*, etc., and for each concatenate of expressions (of L) provable that it holds of it, it is then provable that for all a the given property holds of a.[12]

[12] For a full treatment of this syntax, see *Truth and Denotation*, Chapter III; A. Tarski, *Logic, Semantics, Metamathematics* (Oxford: Clarendon Press, 1956), *passim*; and W. V. Quine, *Mathematical Logic*, pp. 281–306. In iterating concatenation we shall frequently omit inner parentheses. Thus we shall often write '$(a \frown b \frown c)$' in place of either '$((a \frown b) \frown c)$' or '$(a \frown (b \frown c))$'. This is justified in virtue of the associative law of concatenation, which follows from the axioms assumed.

The denotational semantics of the object-language L is to be handled on the basis of some suitable semantical primitive such as designation or denotation. Let us choose denotation in the sense of *multiple denotation*, according to which roughly a one-place predicate denotes severally the objects to which it applies. Let this relation be symbolized by 'Den' in contexts of the form

'a Den x',

which may be read 'the expression a denotes the individual x'.

Where in fact a Den x, a is to be a primitive one-place predicate constant or a one-place abstract containing no free variables. This is to be stipulated by a *semantical rule* or axiom. Also we assume that

$$\vdash a \text{ Den } x \equiv \text{--}x\text{--},$$

where '--x--' is (the translation of) a formula of the object-language containing 'x' as its only free variable (if any) and 'a' is taken as the structural description of the abstract '$x \ni$--x--', or else 'a' is the structural description of a one-place primitive predicate constant and '--x--' is the sentential function consisting of that one-place predicate constant followed by 'x'. It is well known that upon the basis of these two *Rules of Denotation*, presupposing essentially the syntax and logic already outlined, the full denotational semantics of L may be developed.[13]

By way of summary, then, we have the following semantical rules.

DenR1. $\vdash a \text{ Den } x \equiv \text{--}x\text{--}$, if (etc., as given),

DenR2. $\vdash a \text{ Den } x \supset \text{PredConOne } a$,

where 'PredConOne a' expresses that a is a primitive one-place predicate constant or a one-place abstract containing no free variables.

The semantical *truth-concept* for L may immediately be defined in terms of multiple denotation. We let

'Tr a' abbreviate '(Sent a . (Eb)(Vbl b . (x)($b \frown inv ep \frown a$) Den x))'.

In the definiens here (or right-hand side of the definition) 'Sent a' is to express that a is a *sentence* of L and 'Vbl a' that a is a *variable* of L.

[13] See *Truth and Denotation*, Chapters IV and V.

These notions are presupposed as defined in the foregoing syntax. '*invep*' ('inverted epsilon') is to be the primitive structural-descriptive name of '∋' (characterized by the Rule of Abstraction). According to this definition of 'Tr', then, a sentence '-----' of L is true if and only if some (vacuous) abstract 'x∋-----', where x is a variable, denotes *all* objects. On the basis of this definition the requisite properties concerning truth may readily be proved.

The predicate 'Tr' in this semantical metalanguage is said to be *adequate* provided every formula of the metalanguage of the form

'Tr a ≡ -----',

where '-----' is (the translation of) some sentence of L of which 'a' is the structural description, is provable. On the basis provided we can in fact show that 'Tr' is adequate in this sense. Hence we have the following *Adequacy Rule* for 'Tr'.

TC1. ⊢ Tr a ≡ -----, where (etc.).

(Theorems will be numbered throughout a given chapter to indicate the section in which they occur. Thus, *TC1* here is the *first* theorem of §C.)

The foregoing definition of 'Tr' is given in the semantical metalanguage for L based on Den. This metalanguage, it should be noted, contains *two* kinds of variables, one kind ranging over the individuals of L, the other over the expressions of L. Likewise it contains the whole of L (or a translation of such) as a part. The identity sign is now ambiguous, standing for a relation between the (presumably nonlinguistic) entities of L, on the one hand, and for a relation between the expressions of L, on the other. Similarly we have expressions for virtual classes of objects as well as for virtual classes of expressions. These latter will be of especial importance later. Many of the definitions which will be introduced subsequently in fact will be concerned not just with individual expressions of L but with virtual classes of such.

Hereafter we use 'F', 'G', and 'H', possibly with primes or numerical subscripts, to stand for any *expressional abstracts*, i.e., any one-place abstracts of the form

'a∋A',

where ***a*** is any expressional variable and A is any formula of the semantical metalanguage containing ***a*** as its only free variable. These letters thus stand for expressions for virtual classes of expressions of L.

It will be useful to have available some notions of a *Boolean algebra* of virtual classes of expressions. Thus we may introduce notations for the *logical sum* and the *logical product* of two virtual classes of expressions, for the *negation* of such a virtual class, and for the *universal* and *null* virtual classes of expressions. More specifically, we let

$$`(F \cup G)\text{' abbreviate } `a \ni (Fa \vee Ga)\text{'},$$

$$`(F \cap G)\text{' abbreviate } `a \ni (Fa \,.\, Ga)\text{'},$$

$$`-F\text{' abbreviate } `a \ni \sim Fa\text{'},$$

$$`\mathrm{V}\text{' abbreviate } `a \ni a = a\text{'},$$

and

$$`\Lambda\text{' abbreviate } `-\mathrm{V}\text{'}.$$

These definitions introduce, respectively, the Boolean notions mentioned above. Some of these notions will be needed subsequently.

Care should be taken not to confuse the '$\cap$' for the logical product of two virtual classes for the symbol for concatenation introduced above. The usage intended, however, will always be clear from the context.

The *identity* and *inclusion* of virtual classes of expressions may be defined as follows.

$$`F = G\text{' abbreviates } `(a)(Fa \equiv Ga)\text{'},$$

and

$$`F \subset G\text{' abbreviates } `(a)(Fa \supset Ga)\text{'}.$$

Concerning these various notions we have the usual laws of a Boolean algebra, laws of association, commutation, distribution, existence, etc.

We have sharply separated object- and metalanguage here in order to assure consistency. Without such separation it is well-known that contradictions are likely to arise. In explicitly employing the so-called hierarchy of languages as a necessary foundation for the theory of quasi-intensions and intensions, we are merely continuing the work of Carnap, Church, Tarski, and others. And no clear-cut alternative to

this hierarchy, in a form suitable for present purposes, is known to be possible.

D. ANALYTIC TRUTH AND L-SEMANTICS

An *analytic truth* of *L* is a sentence of *L* true *wholly in virtue of its form*. Sometimes the analytic truths are called *logical truths*. Let 'Anlytc *a*' express that the sentence *a* is a logical or analytic truth of *L*. The definition of the predicate 'Anlytc' is intended to explicate this fundamental notion in essentially the senses of Hilbert-Bernays, Tarski, or Carnap.[14] The analytic or logically true sentences include just those sentences true in virtue of their truth-functional or quantificational structure or in virtue of the theory of identity. There is nothing mysterious or vague about this notion when it and its semantical foundations are properly understood.

Roughly speaking, a sentence *a* of *L* is logically or analytically true in *L* if and only if (i) it is true and (ii) every result of replacing simultaneously all occurrences of primitive predicate constants in *a* by primitive predicate constants of the proper degree, or by abstracts of the proper degree containing no free variables, is also true. Of course *L* must be sufficiently rich in its supply of primitive predicate constants and abstracts; otherwise such a definition might fail of its intent. But the *L*'s under consideration are all presumed to satisfy this requirement. The actual definition of 'Anlytc' involves a good deal of technicality and need not be given here.

In terms of 'Anlytc' many notions of Carnap's *L-semantics* are definable, such notions as *L-implication*, *L-disjunction*, *L-falsity*, *F-truth*, *F-falsity*, and the like.[15] A sentence *a* is said to *L-imply* a sentence

[14] See *inter alia* D. Hilbert u. P. Bernays, *Grundlagen der Mathematik*, Vol. I (Berlin: Springer, 1937), p. 8; D. Hilbert u. W. Ackermann, *Grundzüge der Theoretischen Logik*, 4th ed. (Berlin: Springer, 1959), pp. 75–76; A. Tarski, *Logic, Semantics, Metamathematics*, pp. 409–20; R. Carnap, *Introduction to Semantics* (Cambridge: Harvard University Press, 1946), *passim*, and *Logical Foundations of Probability* (Chicago: University of Chicago Press, 1950), pp. 83–84; and E. W. Beth, *The Foundations of Mathematics* (*Studies in Logic and the Foundations of Mathematics*, Amsterdam: North-Holland Publishing Co., 1959), esp. p. 233 ff. Cf. also the author's *The Notion of Analytic Truth* (Philadelphia: University of Pennsylvania Press, and London: Oxford University Press, 1959).

[15] See Carnap, *Introduction to Semantics*, pp. 56–154, *Meaning and Necessity*, pp. 7–16 and *passim*, and *Logical Foundations of Probability*, pp. 82–89.

b, for example, if and only if the sentence (*a hrsh b*) is itself analytic, where '*hrsh*' (Sheffer's 'horseshoe') is the defined structural description of '$\supset$'. '($A \supset B$)', where A and B are formulae, was defined, it will be recalled, as '($\sim A$ v B)', so that '(*a hrsh b*)' may be regarded as an abbreviation for the structural description '($lp \frown tilde \frown a \frown vee \frown b \frown rp$)'. '*lp*' (left parenthesis) is the (primitive) structural description of '(' and '*rp*' (right parenthesis) of ')'.

A sentence *a* is L-false ('LFls *a*') if and only if its negation ($tilde \frown a$) is Anlytc or L-true. A sentence *a* is *synthetic* or F-true ('Synthc *a*') provided it is true but not analytically so.

Let '(*a dot b*)' be an abbreviation for the structural description '($tilde \frown lp \frown tilde \frown a \frown vee \frown tilde \frown b \frown rp$)' and '(*a tripbar b*)' an abbreviation for '((*a hrsh b*) *dot* (*b hrsh a*))'. In effect then we may use '*dot*' as the structural description for '.' and '*tripbar*' ('triple bar') for '$\equiv$'.

A few useful theorems concerning Anlytc and the L-semantical notions are the following.

TD1. $\vdash$ Anlytc $a \supset \sim$Anlytc (*tilde a*).

TD2. $\vdash$ (Sent a . Sent b . (Anlytc a v Anlytc b)) $\supset$ Anlytc (*a vee b*).

TD3. $\vdash$ (Sent a . Sent b) $\supset$ (Anlytc (a dot b) $\equiv$ (Anlytc a . Anlytc b)).

TD4. $\vdash$ (Sent a . Sent b) $\supset$ (Synthc (*a dot b*) $\equiv$ ((Synthc a . Tr b) v (Tr a . Synthc b))).

TD5. $\vdash$ (Anlytc a . Anlytc (*a hrsh b*)) $\supset$ Anlytc b.

TD6. $\vdash$ (Anlytc a . Anlytc (*a tripbar b*)) $\supset$ Anlytc b.

TD7. $\vdash$ (Synthc a . Anlytc (*a tripbar b*)) $\supset$ Synthc b.

Among the analytic truths are to be reckoned all the *logical theorems* of *L* which contain no free variables and hence are sentences, plus all the results of abbreviating such by means of definitions. The logical theorems are those which follow wholly from the logical axioms and rules. In fact we have not only that all logical theorems which are sentences are Anlytc's, but also that all Anlytc's are logical theorems. In a certain sense this circumstance gives us a condition under which

any definition of logical or analytic truth for L might be said to be *adequate*. Let 'LogThm a' express that a is a *logical theorem* of L, the exact definition being presupposed in the underlying syntax. We have then the following law.

TD8. $\vdash \mathrm{Sent}\ a \supset (\mathrm{Anlytc}\ a \equiv \mathrm{LogThm}\ a)$.[16]

In view of this Adequacy Principle, we note that the notion of being a logical theorem is a *syntactical correlate* of the semantical notion of analytic truth. Hence for some purposes we ought to be able to use the one in place of the other. In fact we shall have occasion to do this later. (Cf. (IV, H–I) and (VI, G) below.)

As a few typical examples of analytic sentences (or logical theorems which are sentences) of L, we note the following: '$\sim \mathrm{a} = \mathrm{b}$', '$\sim \mathrm{a} = \mathrm{c}$', '$\sim \mathrm{b} = \mathrm{c}$', '$(x)x = x$', '$(x)(\mathrm{P}x \vee \sim \mathrm{P}x)$', '$(x)\mathrm{P}x \supset \mathrm{Pa}$', '$(x)((\mathrm{P}x \,.\, \mathrm{Q}x) \supset \mathrm{P}x)$', and so on, where 'P' and 'Q' are primitive one-place predicate constants. Included here also are some equivalent formulae in abstracted form, as, for example, '$(x \ni \sim x = \mathrm{b})\,\mathrm{a}$', '$(x \ni \sim \mathrm{a} = x)\,\mathrm{b}$', '$(y \ni (x)\mathrm{P}x \supset \mathrm{P}y)\,\mathrm{a}$', and so on. (For perspicuity, these formulae are written with the additional parentheses enclosing the abstracts.) Also sentences of L such as '$(\imath x.A) = (\imath x.A)$', where A contains 'x' as its only free variable, '$(\imath x.(\mathrm{P}x \,.\, \mathrm{Q}x)) = (\imath y.(\mathrm{Q}y \,.\, \mathrm{P}y))$', and '$(\imath x.x = \mathrm{a}) = \mathrm{a}$' are clearly analytic. But note that a sentence such as '$\mathrm{P}\,(\imath x.(\mathrm{P}x \,.\, \mathrm{Q}x))$' is *not* analytic. If in fact there is not one and only one x having both P and Q, then $(\imath x.(\mathrm{P}x \,.\, \mathrm{Q}x))$ is a*; but whether a* has P or not is not a matter of logic but of fact.

Three of the fundamental notions of semantics, including L-semantics, are the notions of truth (Tr), of being analytic (Anlytc), and of being synthetic or factually true (Synthc). We shall see that these play an especially important role in the theory of intensions. To these we shall wish to add the notion of being a *theorem* of L. The theorems include the logical theorems, and the nonlogical ones as well. In general the truths of L and the theorems of L (which are also sentences,

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[16] This law is essentially Gödel's Completeness Theorem. Cf. K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls," *Monatshefte für Mathematik und Physik* XXXIX (1930): 349–60. Cf. also L. Henkin, "The Completeness of the First-Order Functional Calculus," *The Journal of Symbolic Logic* 14 (1949): 159–66.

i.e., contain no free variables) do not coincide, except for especially restricted *L*, so that it is essential to distinguish them here.[17]

It should be remarked that no distinction has been drawn here between logical and analytic truth. Occasionally a distinction is made between these according to which an analytic truth arises from a logical one by replacing a constant by a synonym. It is then often objected that we have no "clear" notion of being a synonym and hence no "clear" notion of analytic truth.[18] But note that here in the systems *L* we have in effect only a very narrow notion of synonym, according to which a defined expression is synonymous with the expression which defines it. Thus certain sequences of primitive signs of *L* may be replaced by defined signs and conversely. Strictly speaking, all of the expressions of *L* should be written in primitive notation. All defined signs are thus in principle eliminable although in practice it may be highly convenient to use them.

If the reader desires, he may write 'LogTr', for logical truth, in place of 'Anlytc' throughout. Also he may presuppose for it either the definition of 'Anlytc' mentioned above or any of the various alternative definitions given in the recent literature, bearing in mind, however, that not all of these are philosophically above suspicion. And for many of them, metalanguages more powerful than those used here are needed.

An especially useful notion of L-semantics for immediate purposes is that of the *L-equivalence of sentences*. We may let

'a $\mathrm{LEquiv}_{\mathrm{Sent}}$ b' abbreviate '(Sent a . Sent b . Anlytc (a *tripbar* b))'.

Two sentences are L-equivalent according to this definition if and only if the sentence which says (in effect) that they are materially equivalent is itself analytic.

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[17] Cf. K. Gödel, "Uber Formal Unentscheidbare Sätze der *Principia Mathematica* und Verwandter Systeme I," *Monatshefte für Mathematik und Physik* XXXVIII (1931): 173–98.

[18] See esp. Quine's "Two Dogmas of Empiricism," in *From a Logical Point of View*; M. G. White's "The Analytic and the Synthetic: An Untenable Dualism," reprinted in L. Linsky, *Semantics and the Philosophy of Language* (Urbana: University of Illinois Press, 1952); and the author's "On 'Analytic'," *Philosophical Studies* 3 (1952): 42–47.

As examples, we note that the sentences '(Pa . Qa)' and '(Qa . Pa)' of L are L-equivalent. Also '$(x)(\mathrm{P}x \supset \mathrm{Q}x)$' and '$(y)(\mathrm{P}y \supset \mathrm{Q}y)$' and '$(y)({\sim}\mathrm{Q}y \supset {\sim}\mathrm{P}y)$', '$\mathrm{P}(\imath x.(\mathrm{P}x \,.\, \mathrm{Q}x))$' and '$\mathrm{P}(\imath x.(\mathrm{Q}x \,.\, \mathrm{P}x))$', and so on.

L-equivalence for *individual constants* and for *predicate constants* of fixed degree may also be introduced. Two (primitive or defined) individual constants are said to be L-equivalent provided the sentence which says that they are identical is itself analytic.

'a LEquiv$_{\mathrm{InCon}}$ b' abbreviates '(InCon a . InCon b . Anlytc $(a \frown id \frown b)$)'.

'*id*' is the primitive structural description of '$=$' and 'InCon a' states that the expression a is a primitive individual constant of L or a description '$(\imath x.A)$' containing no free variables. Such a description we may occasionally call a *defined* individual constant, as has already been remarked.

We note that each primitive individual constant is L-equivalent with itself but with no other primitive individual constant. A defined individual constant such as '$(\imath x.(\mathrm{P}x \,.\, \mathrm{Q}x))$' is L-equivalent with '$(\imath x.(\mathrm{Q}x \,.\, \mathrm{P}x))$' or with '$(\imath x.{\sim}({\sim}\mathrm{Q}x \vee {\sim}\mathrm{P}x))$', and so on. The constant 'a' is L-equivalent with '$(\imath x.x = \mathrm{a})$'.

Let 'PredConOne a', as above, express within the metalanguage that a is a primitive one-place predicate constant of L or a one-place abstract containing no free variables. Such an abstract may occasionally be called a *defined* one-place predicate constant. Similarly 'PredConTwo a' is to express that a is a primitive two-place predicate constant or a two-place abstract containing no free variables. Also let '*ex*' be the primitive structural description of the individual variable 'x' and let '$(c\ qu\ d)$', where '*qu*' is to suggest 'quantification', be an abbreviation for the structural description '$(lp \frown c \frown rp \frown d)$'. Hence where 'P' and 'Q' are primitive or defined one-place predicate constants of L and 'a' and 'b' are respectively taken as their structural descriptions, '$(ex\ qu\ (a \frown ex\ tripbar\ b \frown ex))$' is the structural description of the sentence of L '$(x)(\mathrm{P}x \equiv \mathrm{Q}x)$'. Recall that new variables of L may be formed from ones already available by concatenating them with an *accent* '$'$'. Let '*ac*' be the structural description of '$'$'. Then '$(ex \frown ac)$' is the structural description of 'x''. And so on. Hence where 'R' and 'S' are primitive or defined two-place predicate constants of L and 'a'

and 'b' are respectively taken as their structural descriptions, the sentence '$(x)(x')(\mathrm{R}xx' \equiv \mathrm{S}xx')$' of L will have the structural description '($ex\ qu\ ex \frown ac\ qu\ (a \frown ex \frown ex \frown ac\ tripbar\ b \frown ex \frown ex \frown ac)$)'.

Two (primitive or defined) one-place predicate constants may now be said to be L-equivalent provided the sentence which says (roughly) that the one holds of every object of which the other holds and conversely is analytic.

'a $\mathrm{LEquiv}_{\mathrm{PredConOne}}$ b' abbreviates '(PredConOne a . PredConOne b . Anlytc ($ex\ qu\ (a \frown ex\ tripbar\ b \frown ex)$))'.

And similarly for two-place predicate constants.

'a $\mathrm{LEquiv}_{\mathrm{PredConTwo}}$ b' abbreviates '(PredConTwo a . PredConTwo b . Anlytc ($ex\ qu\ ex \frown ac\ qu\ (a \frown ex \frown ex \frown ac\ tripbar\ b \frown ex \frown ex \frown ac)$))'.

It might be thought that we are being unnecessarily laborious as to logical detail, that we are being a mere "purveyor of unprofitable subtleties." It seems, however, that a clear-cut foundation for the theory of intensions and decision on a firm foundation in modern syntax and semantics has never been given. Hence the detail seems justified to assure adequacy of formulation. Further, we must use logical notions (the sentential connectives, quantifiers, etc.) in one form or another in all areas of applied logic. Little harm therefore, it would seem, can arise by being explicitly attentive to the details of this use. Surely we cannot know too much about, or be too aware of, notions and methods we are constantly employing.

E. OBJECTIVE QUASI-INTENSIONS

Let us now turn to the theory of intensions. In this chapter we shall be concerned only with what are to be called *quasi*-intensions. All of the quasi-intensions are to be *virtual classes of expressions* of one kind or another. They are not genuine intensions in the sense of being a nonlinguistic kind of entity, an entity independent of language. The virtual classes of expressions considered, on the other hand, are closely akin conceptually to types of entities we should wish to call genuine intensions. Genuine intensions will be introduced in Chapter V. For the present we consider only a few kinds of quasi-intension, but the

whole subject will be discussed more fully in the later chapter, where quasi-intensions and genuine intensions will be interrelated within a systematic framework.

The quasi-intensions here will also be called *objective* (or *semantical*) to distinguish them from the *subjective* (or pragmatical) quasi-intensions to be introduced in Chapter IV. Objective intensions and quasi-intensions in no way depend directly upon time or upon the human (or other) subject who uses the language under discussion. They depend rather, or at least primarily, upon logical and semantical features of the language. And among these features especially important is the notion of analytic truth.

The first type of intension to be considered is based on L-equivalence. We may in fact immediately introduce quasi-intensions as suitable virtual classes of L-equivalent expressions. And we must be careful to distinguish the objective quasi-intensions of individual constants from those of one-place predicate constants, those of one-place predicate constants from those of two-place predicate constants, and so on.

First, the objective quasi-intension *based on L-equivalence* of a primitive or defined individual constant may be regarded as the virtual class of L-equivalent individual constants.

'F ObjLEquivQuasiInt$_{\mathrm{InCon}}$ a' abbreviates '(InCon a . $F = b \ni b$ LEquiv$_{\mathrm{InCon}}$ a)'.

Thus, e.g., we have as members of the ObjLEquivQuasiInt$_{\mathrm{InCon}}$ of the *primitive* InCon 'a' the defined InCon's '$(\gamma x.x = \mathrm{a})$', '$(\gamma x.(y)(y = a \supset y = x))$', '$(\gamma x.y \ni y = \mathrm{a}\ x)$', and so on. Let 'P' and 'Q' again be primitive one-place predicate constants. As members of the ObjLEquivQuasiInt$_{\mathrm{InCon}}$ of the *defined* InCon '$(\gamma x.(\mathrm{P}x\ .\ \mathrm{Q}x))$' we have '$(\gamma x.(\mathrm{Q}x\ .\ \mathrm{P}x))$', '$(\gamma x.\sim(\mathrm{P}x \supset \sim\mathrm{Q}x))$', '$(\gamma y.(\mathrm{P}y\ .\ \mathrm{Q}y\ .\ y = y))$', '$(\gamma z.(\mathrm{P}z\ .\ x \ni (\mathrm{Q}x\ .\ x = x)\ z))$', and so on.

Similarly the objective quasi-intension based on L-equivalence of a primitive or defined one-place *predicate* constant may be regarded as the virtual class of L-equivalent predicate constants.

'F ObjLEquivQuasiInt$_{\mathrm{PredConOne}}$ a' abbreviates '(PredConOne a . $F = b \ni b$ LEquiv$_{\mathrm{PredConOne}}$ a)'.

And similarly for primitive or defined two-place predicate constants, three-place predicate constants, and so on.

Let 'P' and 'Q' again be primitive one-place predicate constants of *L*. As members of the $\mathrm{ObjLEquivQuasiInt}_{\mathrm{PredConOne}}$ of 'P' we have then 'P' itself, '$x \ni (\mathrm{P}x \,.\, x = x)$', '$y \ni (\mathrm{P}y \vee \mathrm{P}y)$', and so on. Let 'U' for the moment be introduced as an abbreviation for, say, '$x \ni (\mathrm{P}x \vee \mathrm{Q}x)$'. Then '$x \ni (\mathrm{Q}x \vee \mathrm{P}x)$', '$y \ni (\sim \mathrm{Q}y \supset \mathrm{P}y)$', '$z \ni (w \ni (\mathrm{Q}w \vee \mathrm{P}w)\, z)$', and so on, are members of the $\mathrm{ObjLEquivQuasiInt}_{\mathrm{PredConOne}}$ of 'U'.

It has frequently been recognized that L-equivalence is a suitable notion for introducing one kind of intension, and quasi-intensions based on L-equivalence seem a very "natural" kind of quasi-intension. Members of one are given by spelling out or enumerating L-equivalent constants, much as (to speak loosely) the meaning of a term in ordinary language is given by spelling out or enumerating the synonyms to be found, say, in the Oxford English Dictionary. We then know how to use the given constant if we know how to use constants L-equivalent with it.

We have immediately theorems of *existence* and *uniqueness*.

TE1. $\vdash F\, \mathrm{ObjLEquivQuasiInt}_{\mathrm{InCon}}\, a \supset (\mathrm{E}b)Fb$.

TE2. $\vdash (F\, \mathrm{ObjLEquivQuasiInt}_{\mathrm{InCon}}\, a \,.\, G\, \mathrm{ObjLEquivQuasiInt}_{\mathrm{InCon}}\, a) \supset F = G$.

TE3. $\vdash F\, \mathrm{ObjLEquivQuasiInt}_{\mathrm{PredConOne}}\, a \supset (\mathrm{E}b)Fb$.

TE4. $\vdash (F\, \mathrm{ObjLEquivQuasiInt}_{\mathrm{PredConOne}}\, a \,.\, G\, \mathrm{ObjLEquivQuasiInt}_{\mathrm{PredConOne}}\, a) \supset F = G$.

And so on for PredConTwo's, etc.

All of these quasi-intensions are virtual classes of linguistic entities, i.e., of expressions of such and such a kind. Such intensions we might therefore call also *expressional* intensions. Expressional intensions might be thought to rely too heavily on purely notational features of the language at hand and therefore should not be regarded as intensions at all. But of course an *interpretation* of the language is presupposed, an interpretation being supplied by the underlying truth-concept involved in the definition of 'Anlytc', which in turn is used in introducing L-equivalence. Strictly then the quasi-intension of a constant regarded as a virtual class of expressions depends fundamentally upon

this interpretation. The objective L-equivalent quasi-intension of an individual constant *a*, for example, is the virtual class of all individual constants L-equivalent to *a in the given interpretation.*[19]

Later, in Chapter V, certain kinds of objective intensions will be introduced as classes of *nonlinguistic* entities. Such genuine intensions might be called *nonexpressional* and should enjoy some real advantages over expressional or quasi-intensions, as we shall see. But they will be defined only within a more powerful kind of metalanguage, involving the theory of types.

Over virtual classes of expressions in general we have no quantifiers. Quantifiers over the various kinds of expressional or quasi-intensions may be introduced by definition, however, as follows.

First we note that we cannot introduce the universal quantifier over, for example, objective L-equivalent quasi-intensions for individual constants *simpliciter* by means of a formula of the form

$$\text{`}(a)(F\ \mathrm{ObjLEquivQuasiInt}_{\mathrm{InCon}}\ a \supset \text{---}F\text{---})\text{'},$$

where '---*F*---' is some formula containing '*F*' in the appropriate way. This kind of formula clearly does not yield what we wish. But suppose we let

$$\text{`}\mathrm{objLEquivQuasiInt}_{\mathrm{InCon}}\ a\text{'} \text{ abbreviate } \text{`}b \ni b\ \mathrm{LEquiv}_{\mathrm{InCon}}\ a\text{'}.$$

The definiendum (left-hand side of the definition) here contains '$\mathrm{objLEquivQuasiInt}_{\mathrm{InCon}}$' not as a predicate but as a *functor* (name of a function), and hence the first letter is not capitalized (in accord with a convention frequently employed). The whole definiendum may be read 'the objective L-equivalent quasi-intension (of the kind appropriate to individual constants) of *a*'. If in fact *a* is an individual constant, the definiendum here gives us a name for the $\mathrm{objLEquivQuasiInt}_{\mathrm{InCon}}$ of *a*. If *a* is not an individual constant, this definiendum gives us a name merely of the null virtual class of expressions.

Universal quantification over objective L-equivalent quasi-intensions of the kind appropriate to individual constants may then be achieved by the formula (of the metalanguage)

$$\text{`}(a)(\mathrm{InCon}\ a \supset \text{---}\mathrm{objLEquivQuasiInt}_{\mathrm{InCon}}\ a\text{---})\text{'},$$

where '---$\mathrm{objLEquivQuasiInt}_{\mathrm{InCon}}\ a$---' is a formula containing

[19] Cf. *Truth and Denotation*, pp. 278–81, for an analogous argument.

'$\text{objLEquivQuasiInt}_{\text{InCon}}\ a$' in the proper (grammatical) way. Similarly existential quantification may be achieved by a formula of the form

$$\text{'}(\text{E}a)(\text{InCon}\ a\ .\ \text{---objLEquivQuasiInt}_{\text{InCon}}\ a\text{---})\text{'}.$$

If a is an InCon then and only then is its objective L-equivalent quasi-intension nonnull.

TE5. $\vdash \text{InCon}\ a \equiv {\sim}\text{objLEquivQuasiInt}_{\text{InCon}}\ a = \Lambda.$

Also quantifiers over the other types of quasi-intensions defined may be introduced by means of appropriate functors. Each such functor corresponds to the name of the corresponding relation but is not capitalized. Thus '$\text{objLEquivQuasiInt}_{\text{PredConOne}}\ a$' is to be the expression for the objective L-equivalent quasi-intension (of the type appropriate to one-place predicate constants) of a. And so on. Clearly we have that

TE6. $\vdash \text{PredConOne}\ a \equiv {\sim}\text{objLEquivQuasiInt}_{\text{PredConOne}}\ a = \Lambda.$

And so on.

Of course quasi-intensions here are wholly determined by their terms. Having an at most denumerable number of terms (constants) we have an at most denumerable number of quasi-intensions of each kind.

It is assumed throughout that we are committed by L to the existence of the entities over which the variables of L range. Let these entities constitute a domain or sort D. Then the existential quantifier of L carries with it the real force of existence-in-D. (Note that we speak here of *existence-in-D*, not of existence in any wider sense.[20]) The convenience of having quantifiers, both universal and existential, over quasi-intensions is achieved by the foregoing definitions. But quasi-intensions are not assumed therewith to constitute a separate realm of entities *sui generis*. The quantifiers over them are defined in context by means of quantifiers over the corresponding terms. We are thus not committed by these quantifiers to the independent existence of a realm of quasi-intensions. (Cf. (VI, D) below.)

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[20] Cf. the author's "Existential Quantification and the 'Regimentation' of Ordinary Language," *Mind* LXXI, No. 284 (1962): 525–29.

It should be noted that, in using virtual classes throughout, we achieve the convenience of a notation for classes but avoid being committed to their existence. This is of course an extremely important philosophical point. Later, in Chapter V, we shall commit ourselves explicitly to the existence of classes in the metalanguage, and indeed this will be essential for the theory of genuine intensions. For the theory of quasi-intensions, however, the interesting fact is that no such commitment need be made.

The quasi-intensions introduced here are all based on L-equivalence. Several further types, not based on L-equivalence, will be introduced as we proceed.

F. QUASI-PROPOSITIONS

We may now introduce also a semantical notion of *quasi-proposition.* Propositions are sometimes regarded as classes of synonymous or L-equivalent sentences.[21] Propositions so regarded are thus merely quasi-propositions in the sense of 'quasi' being used here. Having no real classes in the metalanguage, we must regard the classes involved as merely virtual.

Thus the *quasi-proposition based on L-equivalence* corresponding to a sentence a may be regarded as the virtual class of sentences L-equivalent with a. We may let

'F QuasiProp a' abbreviate '(Sent a . $F = b \ni b$ LEquiv$_{\text{Sent}}$ a)'.

We have then existence and uniqueness laws analogous to *TE1–TE2* above.

Also if desired we may introduce a functor for quasi-propositions based on L-equivalence.

'quasiProp a' abbreviates '$b \ni b$ LEquiv$_{\text{Sent}}$ a'.

'quasiProp a' as thus defined is always significant. If a is not a sentence, however, quasiProp a becomes the null virtual class of expressions.

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[21] Cf., e.g., B. Russell, *An Inquiry into Meaning and Truth* (New York: Norton, 1940), p. 209; R. Carnap, *Meaning and Necessity*, p. 152; and W. V. Quine, "Notes on Existence and Necessity," *The Journal of Philosophy* 40 (1943): p. 120 (reprinted in Linsky).

TF1. $\vdash \text{Sent}\ a \equiv \sim\text{quasiProp}\ a = \Lambda$,

TF2. $\vdash \text{quasiProp}\ a = \Lambda \equiv \text{quasiProp}\ (\textit{tilde}\ a) = \Lambda$,

and

TF3. $\vdash ((\text{quasiProp}\ a = \Lambda\ .\ \sim\text{quasiProp}\ b = \Lambda) \vee (\sim\text{quasiProp}\ a = \Lambda\ .\ \text{quasiProp}\ b = \Lambda)) \supset \text{quasiProp}\ (a\ \textit{vee}\ b) = \Lambda$.

But if quasiProp (a *vee* b) = Λ then at least one of the quasi-propositions corresponding to a or b must be null.

TF4. $\vdash \text{quasiProp}\ (a\ \textit{vee}\ b) = \Lambda \supset (\text{quasiProp}\ a = \Lambda \vee \text{quasiProp}\ b = \Lambda)$.

We achieve the effect, and hence the convenience, of quantification over quasi-propositions by a method similar to that of the preceding section. To say that such and such holds of quasiProp a for all sentences a, is in effect to say that such and such holds of all quasi-propositions. In general if '---quasiProp a---' is some formula of the metalanguage containing 'quasiProp a' in one or more places,

$$\text{`}(a)(\text{Sent}\ a \supset \text{---quasiProp}\ a\text{---})\text{'}$$

and

$$\text{`}(\text{E}a)(\text{Sent}\ a\ .\ \text{---quasiProp}\ a\text{---})\text{'}$$

enable us to achieve the effect of universal and existential quantifiers respectively over quasi-propositions.

Quasi-propositions are of course virtual classes of linguistic objects and are relative to a given language-system. This is therefore a somewhat restricted notion. We speak of a quasi-proposition only *of* a given sentence, as it were, *of* a given language-system. Frequently it is maintained that propositions are a kind of nonlinguistic object independent of language. Propositions in this sense, as abstract objects *sui generis*, cannot be handled in the present kind of metalanguage. (Cf. (VI, F) below.)

Let a be the sentence '$(x)(\text{P}x \supset \text{Q}x)$' of L, where 'P' and 'Q' are primitive one-place predicate constants. As members of the quasiProp of a we have then all L-equivalent sentences, sentences such as '$(x)(\sim\text{Q}x \supset \sim\text{P}x)$', '$(y)(\sim\text{P}y \vee (\text{Q}y\ .\ y = y))$', and so on.

Closely related with quasi-propositions are two other notions, involving not L-equivalence but L-implication. Consider the virtual class of all sentences which a given sentence L-implies. This class constitutes what we may call the *L-content* of the given sentence.[22] Similarly, the *L-range* of a given sentence may be regarded as the virtual class of all sentences which L-imply it.[23] The sentence itself is then a member of both its L-content and its L-range. Clearly the quasi-proposition, the L-content, and the L-range of a given sentence are exceedingly important virtual classes of expressions. They should be of especial interest for the philosopher interested in the complete logical analysis of a given sentence. In stipulating the three we in effect give an exhaustive description of one aspect of its logical character. Such description is after all much of the aim of philosophic analysis conceived as the "logical geography" of concepts.[24]

G. SOME FURTHER QUASI-INTENSIONS

There are several further types of objective quasi-intensions to be distinguished from the foregoing, which are of interest on their own account and are definable within the semantical metalanguage here.

Let '$a \ \mathrm{S}_c^b \ d$' express within the underlying syntax that the *sentence a* differs from the *formula d* only in containing the primitive or defined *individual constant b* wherever there are free occurrences of the *variable c* in *d*, where *c* has at least one free occurrence in *d* and *d* contains no other free variables. The definition of this notion presents no difficulty and may be given within the syntax described in §C above. As an example, let *d* be '(P*x* v (*x*)Q*x*)', *c* be '*x*', and *b* the individual constant 'a'. The expression *a* is then '(Pa v (*x*)Q*x*)' where $a \ \mathrm{S}_c^b \ d$ or $a \ \mathrm{S}_c^{ay} \ d$ ('*ay*' being the primitive structural description of 'a'). Note that the italic '*a*' is an expressional variable whereas the roman 'a' is an

[22] This notion is closely related to but differs in important technical respects from that suggested by Popper. Cf. his *The Logic of Scientific Discovery* (London: Routledge and Kegan Paul, 1959).

[23] This notion likewise is related to but differs in important technical respects from that suggested by Carnap in his *Introduction to Semantics*, p. 118 ff.

[24] Cf. G. Ryle, "Philosophical Arguments," Inaugural Lecture (Oxford: Oxford University Press, 1945).

individual constant. The 'S' is to suggest 'substitution', so that '$a\ \mathrm{S}_c^b\ d$' may read loosely 'a results from d by substituting b for c'.

Let a now be an individual constant. The class of those properties which the individual designated by a possesses analytically, so to speak, has close kinship with (one kind of) an intension of a. Within the metalanguage here we cannot speak of classes of properties. We can however speak of *virtual classes of the abstracts or predicate constants standing for those properties.* Consider the virtual class of abstracts of the form $(d \frown invep \frown e)$ where $b\ \mathrm{S}_d^a\ e$ and b is an Anlytc. Such a virtual class in effect fully "mirrors" the class of properties mentioned. Hence we may regard it a quasi-intension of the InCon a, more precisely, the objective *analytic* quasi-intension of a. We let

'F ObjAnlytcQuasiInt$_{\mathrm{InCon}}$ a' abbreviate '(InCon a . $F = c \ni (\mathrm{E}b)$ $(\mathrm{E}d)(\mathrm{E}e)(c = (d \frown invep \frown e)\ .\ b\ \mathrm{S}_d^a\ e$. Anlytc b))'.

Let us consider an example; more particularly, let us consider the objective analytic quasi-intension of the primitive individual constant 'a'. It is analytic that the individual a is not identical with the individual b. Let the expression b be the sentence '$\sim \mathrm{a} = \mathrm{b}$'. Let d be a variable, say 'x', and e the formula '$\sim x = \mathrm{b}$'. Then $b\ \mathrm{S}_d^{ay}\ e$ (where, we recall, 'ay' is the structural description of 'a'). Consider the abstract '$x \ni \sim x = \mathrm{b}$'. This expresses what we call loosely an 'analytic property' of the individual a, and hence is a member of the objective analytic quasi-intension of 'a'. For similar reasons '$x \ni (\mathrm{P}x \vee \sim \mathrm{P}x)$' is a member of the objective analytic quasi-intension of 'a'. And so on.

Note that we could equally well write the clause concerning 'F' in the definiens here as '$F = c \ni$(PredConOne c . Anlytc $(c \frown a)$)', in view of the Rule of Abstraction.

The use of 'property' here, and occasionally later, is a mere convenience. No distinction between properties and classes is intended. By 'property' throughout is meant merely 'property-in-extension'.

Roughly speaking, the objective analytic quasi-intension of a primitive or defined individual constant consists of all the abstracts which apply analytically to the individual designated by that constant. Such abstracts surely shed some light upon both the correct use and meaning of that constant. We are thus, to some extent at least, justified in calling the virtual class of all of them an *intension* of such and such a

kind. It might be argued against this notion, however, that the abstracts involved are rather restricted. They are of three main kinds, those standing for the virtual class whose only member is the individual designated by the constant in question, those standing for virtual classes of individuals distinct from a given individual, and those standing for the virtual class of all individuals. Once these are stipulated the entire objective analytic quasi-intension is given. But surely the point remains that these shed *some* light upon the correct use and meaning of the given constant. On the other hand, the objective analytic quasi-intension of an individual constant is probably of less interest than the other kinds of quasi-intension for individual constants to be introduced. Also quasi-intensions for other types of constants have more variety in their membership. Hence it may be that such types are of greater interest in helping to illumine the correct use and meaning of the kind of constant involved.

Some related notions are as follows.

Suppose we replace 'Anlytc b' by 'Tr b' in the definiens of the definition just given. We gain then another interesting class of abstracts, which also may be regarded as a kind of quasi-intension. Let us call it an objective *veridical* quasi-intension. More precisely, we let

'$F\ \mathrm{ObjVerQuasiInt}_{\mathrm{InCon}}\ a$' abbreviate '$(\mathrm{InCon}\ a\ .\ F = c \ni (\mathrm{E}b)\ (\mathrm{E}d)\ (\mathrm{E}e)(c = (d \cap \mathit{invep} \cap e)\ .\ b\ \mathrm{S}_d^a\ e\ .\ \mathrm{Tr}\ b))$'.

The objective veridical quasi-intension of an individual constant a then consists of just those abstracts standing for (to speak loosely) those properties *truly* possessed by the individual designated by a.

Two further closely related notions are gained by replacing 'Tr b' in the definiens here by 'Synthc b' or by 'Thm b' (where 'Thm b' expresses that b is a *theorem* of L). The resulting virtual classes may be called respectively the objective *synthetic* quasi-intension ('$\mathrm{ObjSynthcQuasiInt}_{\mathrm{InCon}}$') and the objective *theoremic* quasi-intension ('$\mathrm{ObjThmQuasiInt}_{\mathrm{InCon}}$') of a.

Let us consider again the primitive InCon 'a'. Let '--a--' be any truth of L containing 'a'. The corresponding abstract '$x \ni$--x--', where '--a--' is a formula differing from '--x--' only in containing 'a' wherever 'x' occurs freely in '--x--', is then a member of the objective veridical quasi-intension of 'a'. If, on the other hand, '--a--' is either a synthetic

sentence or a theorem of L, then '$x \ni$--x--' is a member respectively of the objective synthetic or of the objective theoremic quasi-intension of 'a'. And similarly for defined InCon's.

Suppose L contains 'Socrates' as an additional primitive InCon, and also such predicates (or abbreviations for them) as 'Greek philosopher', 'husband of Xantippe', 'drank the hemlock', and so on. As members of the $\text{ObjAnlytcQuasiInt}_{\text{InCon}}$ of 'Socrates' we have '$x \ni \sim x = \text{a}$', '$x \ni \sim x = \text{b}$', and so on. The predicates 'Greek philosopher', 'husband of Xantippe', 'drank the hemlock' are then members of both the $\text{ObjVerQuasiInt}_{\text{InCon}}$ and the $\text{ObjSynthcQuasiInt}_{\text{InCon}}$ of 'Socrates'. The members of the $\text{ObjThmQuasiInt}_{\text{InCon}}$ could be specified only on the basis of the nonlogical axioms of L, but presumably would be such that all of the predicates and abstracts mentioned would be members of it.

Existence and uniqueness laws analogous to *TE1–TE2* above may be proven for all of these notions of quasi-intension. (The existence law for '$\text{ObjSynthcQuasiInt}_{\text{InCon}}$' requires the hypothesis that there is at least one synthetically true sentence in L containing the individual constant concerned.) Also clearly

TG1. $\vdash (F \ \text{ObjAnlytcQuasiInt}_{\text{InCon}}\ a \,.\, G \ \text{ObjVerQuasiInt}_{\text{InCon}}\ a) \supset F \subset G.$

TG2. $\vdash (F \ \text{ObjAnlytcQuasiInt}_{\text{InCon}}\ a \,.\, G \ \text{ObjThmQuasiInt}_{\text{InCon}}\ a) \supset F \subset G.$

TG3. $\vdash (F \ \text{ObjThmQuasiInt}_{\text{InCon}}\ a \,.\, G \ \text{ObjVerQuasiInt}_{\text{InCon}}\ a) \supset F \subset G.$

TG4. $\vdash (F \ \text{ObjAnlytcQuasiInt}_{\text{InCon}}\ a \,.\, G \ \text{ObjVerQuasiInt}_{\text{InCon}}\ a \,.\, H \ \text{ObjSynthcQuasiInt}_{\text{InCon}}\ a) \supset (G = (F \cup H) \,.\, (F \cap H) = \Lambda).$

When we turn to predicate constants, we note that there are no objective analytic and other quasi-intensions analogous to the kinds defined. There are, however, types of objective analytic and other quasi-intensions appropriate to one-place predicate constants, to two-place predicate constants, etc. These might be called *comprehensional* intensions. (See also (V, B) below.) Sometimes a class or property is said to *comprehend* another class or property provided that the other

is included in it or is a subclass of it. The class of all properties comprehended by a given property is traditionally akin to a type of intension. And the comprehending may be analytic, veridical, synthetic, or theoremic. In an analogous way we may speak of the comprehension of two-place predicate constants, etc. Using these somewhat vague comments as guides, we may frame exact definitions as follows.

We may let

'F ObjAnlytcQuasiInt$_{\text{PredConOne}}$ a' abbreviate '(PredConOne a . F $= c \ni (\mathrm{E}b)$(PredConOne c . Vbl b . Anlytc $(b\ qu\ (a \cap b\ hrsh\ c \cap b))))$'.

A virtual class is said to be the objective analytic quasi-intension based on comprehension of the one-place predicate constant a if it is the virtual class of all one-place predicate constants standing for virtual classes which analytically comprehend the virtual class for which a stands. Note that '$(b\ qu\ (a \cap b\ hrsh\ c \cap b))$' here is the structural description of some sentence of the form

$$`(\boldsymbol{x})(\text{---}\boldsymbol{x} \supset \cdots \boldsymbol{x})'$$

where $\boldsymbol{x}$ is a variable and '---' and ' $\cdots$ ' are PredConOne's.

Let us consider an example. More particularly, let us consider the objective analytic quasi-intension of a one-place primitive predicate constant 'P'. Clearly '$(x)(\mathrm{P}x \supset \mathrm{P}x)$', '$(x)(\mathrm{P}x \supset (\mathrm{P}x \vee \mathrm{Q}x))$' where 'Q' is another primitive one-place predicate constant, and '$(x)(\mathrm{P}x \supset x = x)$' are all analytic. Hence 'P' itself is a member of the objective analytic quasi-intension of 'P'. Also, in view of *Abst*, are '$x \ni (\mathrm{P}x \vee \mathrm{Q}x)$' and '$x \ni x = x$'. A virtual class is then the ObjAnlytcQuasiInt$_{\text{PredConOne}}$ of 'P' provided it contains as members *all* such PredConOne's.

In a related way, objective veridical, synthetic, and theoremic quasi-intensions based on comprehension for one-place predicate constants may be introduced. The relations involved may be symbolized by 'ObjVerQuasiInt$_{\text{PredConOne}}$', etc., in obvious fashion.

Suppose '$(x)(\mathrm{P}x \supset \text{--}x\text{--})$' now is a truth of L. Then '$x \ni \text{--}x\text{--}$' is a member of the ObjVerQuasiInt$_{\text{PredConOne}}$ of 'P'. Being a truth it is either analytic or synthetic, and the abstract mentioned is thus also a member of either the ObjAnlytcQuasiInt$_{\text{PredConOne}}$ or the ObjSynthcQuasiInt$_{\text{PredConOne}}$ of 'P'. If '$(x)(\mathrm{P}x \supset \text{--}x\text{--})$' is also

a theorem of L, the abstract '$x \ni$--x--' is then a member of the $\text{ObjThmQuasiInt}_{\text{PredConOne}}$ of 'P' as well.

To be somewhat more specific, suppose for the moment that the values for variables of L are concrete objects in some sense. Let 'Rx' express that x is a rational being and 'Ax' that x is an animal. We may then introduce

$$\text{'M' as short for '}x \ni (\text{R}x \,.\, \text{A}x)\text{'.}$$

For convenience we may read 'Mx' as 'x is a man', but we do not therewith claim that we have defined the *English* word 'man'. We have merely introduced an approximative, informal reading for 'M'. Clearly 'R', 'A', and 'M' are members of the $\text{ObjVerQuasiInt}_{\text{PredConOne}}$, the $\text{ObjAnlytcQuasiInt}_{\text{PredConOne}}$, and the $\text{ObjThmQuasiInt}_{\text{PredConOne}}$ of 'M'. If 'Tx' expresses that x is two-legged, then 'T' is a member of the $\text{ObjVerQuasiInt}_{\text{PredConOne}}$ and of the $\text{ObjSynthcQuasiInt}_{\text{PredConOne}}$ of 'M' but not of the $\text{ObjAnlytcQuasiInt}_{\text{PredConOne}}$. Whether 'T' is a member of the $\text{ObjThmQuasiInt}_{\text{PredConOne}}$ of 'M' rests clearly upon whether

$$\text{'}(x)(\text{M}x \supset \text{T}x)\text{'}$$

is a theorem of L or not. Much depends here of course upon our presupposing the celebrated definition *homo animal rationalis*. If 'M' is introduced in some way other than by that definition, e.g., as a primitive PredConOne, the situation is quite different. If

(1) $$\text{'}(x)(\text{M}x \equiv (\text{R}x \,.\, \text{A}x))\text{'}$$

is Tr in L, then 'R', 'A', and 'M' are members of the $\text{ObjVerQuasiInt}_{\text{PredConOne}}$ of 'M'. If this sentence moreover is a theorem of L, clearly 'R', 'A', and 'M' are members of the $\text{ObjThmQuasiInt}_{\text{PredConOne}}$. (1) cannot, however, be an *analytic* sentence of L, if 'M', 'R', and 'A' are *primitive* PredConOne's (for the simple reason that one can render the sentence false by a suitable replacement for 'M', namely by '$y \ni \sim(\text{R}y \,.\, \text{A}y)$'). Hence 'R' and 'A' cannot be members of the $\text{ObjAnlytcQuasiInt}_{\text{PredConOne}}$ of 'M'. If (1) is Tr, however, it is then Synthc and both 'R' and 'A' are members of the $\text{ObjSynthcQuasiInt}_{\text{PredConOne}}$.

It would seem that the $\text{ObjAnlytcQuasiInt}_{\text{PredConOne}}$'s here are more interesting than the $\text{ObjAnlytcQuasiInt}_{\text{InCon}}$'s. The reason is that

we have great freedom in the matter of introducing new predicate constants by definition. Abstracts containing no free variables are regarded as defined predicate constants. Let 'E' for the moment be some defined predicate constant. Ordinarily, then, there is a great variety of Anlytc's of the form

$$\text{`}(x)(\mathrm{E}x \supset \text{--}x\text{--})\text{'}.$$

Hence the $\mathrm{ObjAnlytcQuasiInt}_{\mathrm{PredConOne}}$ of 'E' may have an interesting membership. The situation with defined *individual* constants is quite different. There is not too great a variety of Anlytc's of the form

$$\text{`}G\ (\mathit{7}x.A)\text{'},$$

as we have already observed, although there are some, e.g., '$x \ni x = \mathrm{a}\ (\mathit{7}x.x = \mathrm{a})$'.

There is a close kinship between the $\mathrm{ObjAnlytcQuasiInt}_{\mathrm{PredConOne}}$ and what we might call (following Popper) the *L-content* of a PredConOne. The L-content of a sentence, we recall, is the virtual class of all sentences which it L-implies. The L-content of a PredConOne may be introduced as the virtual class of all PredConOne's in which it is analytically contained. In a similar fashion the *L-range* of a PredConOne may be introduced as the virtual class of all PredConOne's which are analytically contained in it.

Note incidentally that although we have spoken informally of an InCon as *designating* an individual, the relation of designation involved has not been formally defined within the semantical metalanguage. But we may, if we wish, let

'a Des x' abbreviate '(InCon a . (Eb)(Vbl b . $(b \frown \mathit{inrep} \frown b \frown \mathit{id} \frown a)$ Den x))'.

A primitive or defined individual constant then designates an object x provided some suitable abstract denotes it.

Some of the distinctions drawn here may appear to the reader as rather tedious. But where exact logical and philosophical matters are concerned, progress in "dispersing the fog" often rests upon making fine distinctions heretofore neglected. And once made, they come to seem as natural as distinctions already familiar.

With the necessary changes, the corresponding kinds of quasi-intensions based on comprehension for two-place predicate constants may be introduced. The relations involved may be symbolized by '$\mathrm{ObjAnlytcQuasiInt}_{\mathrm{PredConTwo}}$', etc. And similarly for three-place predicate constants, etc.

Quantification over each of these types of quasi-intension may be achieved by methods similar to those in §E and §F.

These various types of quasi-intensions are mentioned here only in passing. In Chapters V and VI there will be further discussion of them and of their relationship to other types of intension.

Perhaps the terminology used for some of these notions is not the best or most suggestive. Most of them seem never to have been discussed or even mentioned in the literature heretofore, and new names for them are therefore appropriate. In each case we have tried to use a suggestive name that seems apt.

Let us turn now to another area of semantics, which likewise will be useful in the sequel.

H. INDUCTIVE LOGIC

In his *Logical Foundations of Probability* Carnap has formulated the beginnings of a theory of *degree of confirmation* or of an *inductive logic.* The object-languages Carnap considers are in essential respects like the *L*'s above. The sentences of such systems are subjected to a detailed analysis, as a result of which a numerical *measure* is assigned to them. Of course such a measure may be assigned in many different ways. Certain of these ways presumably lead to a "reasonable" concept of degree of confirmation or of *evidential support.*

Let *a* and *b* be any two sentences of *L*. The degree of confirmation of the hypothesis *a* on the evidence *b*, symbolized by 'c(*a*, *b*)', is to be the measure of evidential support which *b* gives to *a*. Such a notion is often referred to as a notion of *inductive probability*. And in fact the connection between the notion of degree of confirmation and the principles of inductive probability and inference is a close one. An inductive logic, according to Carnap, comes directly out of a suitably developed theory of degree of confirmation.

In subsequent chapters implicit use is made of what is essentially a

notion of degree of confirmation. Such a notion may be taken to provide implicitly the necessary numerical foothold for the quantitative pragmatics and theory of *subjective* intensions to be developed. But inductive logic will play no explicit role in that theory. In other words, the quantitative pragmatics need not contain the modes of expression needed for a theory of degree of confirmation. But there will be frequent occasion to refer to the inductive logic of L informally. We may wish to compare certain kinds of patterns with corresponding patterns based upon degrees of confirmation. The logically "correct" degrees of confirmation may guide us in framing various definitions and in discovering theorems. But this will not entail that expressions for degrees of confirmation explicitly occur within the pragmatical metalanguage. Indeed, they may so occur if the metalanguage has sufficient power. But we shall not in general presuppose this.

We choose Carnap's theory here not because it is the only theory of confirmation available or even the most satisfactory. But from a logical point of view, more particularly, from the point of view of the formal syntax and semantics above, it appears to be the most carefully developed treatment of confirmation yet put forward. Perhaps some alternative theory could be used equally well, but not without a clear formulation of the syntax and semantics presupposed.

For Carnap, inductive logic or the theory of degree of confirmation consists, roughly speaking, of a denotational semantics augmented by a metric. Carnap's object-language L_n contains just n distinct individual constants as primitive, where 'n' is a *constant* standing for some fixed finite number. L_∞, on the other hand, contains a denumerable infinity of distinct individual constants as primitives. For these systems a syntax and semantics are provided, to which are added numerical functions on sentences taking real numbers between 0 and 1 inclusively as values. Let 'm' for the moment stand for any such function. If a is a sentence of L, m(a) is then a real number assigned to a in some suitable fashion. On the basis of such a metric, a notion of degree of confirmation may be defined. The elementary arithmetic of real numbers must, then, be presupposed.

Suppose now that such a metric m is available for the sentences of the finite languages L_n and that it has certain desirable properties. (These need not be enumerated, but some of them are given by the

theorems *TH1–TH5* below.) A numerical function on two sentences a and b of L_n is then a *confirmation function* if and only if $\mathrm{m}(b) \neq 0$ and

$$\mathrm{c}(a, b) = \mathrm{m}(a\ dot\ b)/\mathrm{m}(b).$$

('$(a\ dot\ b)$', we recall, is the abbreviation for the structural description '$(tilde \frown lp \frown tilde \frown a \frown vee \frown tilde \frown b \frown rp)$'.)

Out of the multiplicity of measure and therewith confirmation functions, some are more suitable for an inductive logic than others. Part of the task of formulating an inductive logic is that of choosing the one or more such functions best suited for a given scientific task.

The measure functions for the sentences of the infinite language L_∞ may be introduced as follows. Let ${}_1\mathrm{m}$, ${}_2\mathrm{m}$, etc., be a sequence of suitable measure functions for the sentences of L_1, L_2, etc., respectively. Then m is said to be the measure function for the sentences in L_∞ corresponding to this sequence if and only if for every sentence a in L_∞ for which such a limit exists,

$$\mathrm{m}(a) = \lim_{n \to \infty} {}_n\mathrm{m}(a); \tag{1}$$

if there is no such limit, $\mathrm{m}(a)$ has no value.

In a similar way suitable confirmation functions for L_∞ may be introduced. Let ${}_1\mathrm{c}$, ${}_2\mathrm{c}$, etc., be a sequence of suitable confirmation functions for L_1, L_2, etc., respectively. c is then said to be a confirmation function for L_∞ corresponding to this sequence if and only if for any sentences a and b of L_∞ for which the limit exists,

$$\mathrm{c}(a, b) = \lim_{n \to \infty} {}_n\mathrm{c}(a, b); \tag{2}$$

if the limit does not exist, c has no value for the arguments a and b.

It should be noted that in the strict mathematical sense expressions such as (1) and (2) are meaningless. Expressions such as '$\lim_{n \to \infty} \mathrm{f}(n)$' are significant only where 'n' is a *variable* not a constant. Carnap's definitions, however, may easily be reformulated to avoid this difficulty.[25]

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[25] See the author's "A Formalization of Inductive Logic," *The Journal of Symbolic Logic* 23 (1958): 251–56.

A few useful theorems, to which reference may subsequently be made, are as follows. Here m and c are to be regarded as any suitable measure or confirmation functions respectively.

TH1. $\vdash$ (Sent a . Sent b) $\supset$ (m(a) $>$ m(b) $\equiv$ m(*tilde* b) $>$ m(*tilde* a)).

TH2. $\vdash$ (Sent a . Sent b) $\supset$ (m(a) $\leqq$ m(a *vee* b) . m(b) $\leqq$ m(a *vee* b)).

TH3. $\vdash$ (Sent a . Sent b) $\supset$ (m(a *dot* b) $\leqq$ m(a) . m(a *dot* b) $\leqq$ m(b)).

TH4. $\vdash$ (Sent a . Sent b . Sent c . (m(a) $>$ m(c) v m(b) $>$ m(c))) $\supset$ m(a *vee* b) $>$ m(c).

TH5. $\vdash$ (Sent a . Sent b . Sent c . m(a *vee* b) $<$ m(c)) $\supset$ (m(a) $<$ m(c) . m(b) $<$ m(c)).

TH6. $\vdash$ (Sent a . Sent b . Sent c . $\sim$LFls c) $\supset$ (c(a, c) $>$ c(b, c) $\equiv$ c(*tilde* b, c) $>$ c(*tilde* a, c)).

TH7. $\vdash$ (Sent a . Sent b . Sent c . Sent d . $\sim$LFls d . (c(a, d) $>$ c(c, d) v c(b, d) $>$ c(c, d))) $\supset$ c((a *vee* b), d) $>$ c(c, d).

TH8. $\vdash$ (Sent a . Sent b . Sent c . Sent d . $\sim$LFls d . c((a *vee* b), d) $<$ c(c, d)) $\supset$ (c(a, d) $<$ c(c, d) . c(b, d) $<$ c(c, d)).

TH9. $\vdash$ (Sent a . Sent b . Sent c . $\sim$LFls c) $\supset$ (c(a, c) $\leqq$ c((a *vee* b), c) . c(b, c) $\leqq$ c((a *vee* b), c)).

TH10. $\vdash$ (Sent a . Sent b . Sent c . $\sim$LFls c) $\supset$ (c((a *dot* b), c) $\leqq$ c(a, c) . c((a *dot* b), c) $\leqq$ c(b, c)).

TH11. $\vdash$ (Sent a . Sent b . Sent c . $\sim$LFls c) $\supset$ c((a *vee* b), c) $=$ (c(a, c) $+$ c(b, c) $-$ c((a *dot* b), c)).

TH12. $\vdash$ (Sent a . Sent b . Sent c . $\sim$LFls c) $\supset$ c((a *dot* b), c) $=$ (c(a, c) $+$ c(b, c) $-$ c((a *vee* b), c)).

TH13. $\vdash$ (Sent a . LogThm a . Sent b . $\sim$LFls b) $\supset$ c(a, b) $=$ 1.

This very rough sketch of Carnap's confirmation theory is of course highly incomplete. But enough has been said to illustrate some of the salient features of the theory, in fact all that will be needed for subsequent purposes.

II

PREFERENCE

As a preliminary to the formulation of a quantitative pragmatics, including the theory of subjective intensions, we consider in this chapter a relation of *preference* as between sentences of a language-system *L*. Ordinarily preference is regarded as a relation not between sentences but between economic goods, commodities, or the like. Of the various relations of preference (for we must recognize that there are many), it may well be that the one between sentences is the fundamental one and that the others can be suitably defined in terms of it. At any event, if we take preference as a relation between sentences and reflect upon the logical syntax and semantics of *L*, an interesting new kind of theory emerges.

How are preferences related to negation? Are there kinds of preferences which accord with the logically correct use of '∼'? How are preferences related to disjunction? Can we find rankings of preferences which accord with the logically correct use of 'v'? The theory of preferences between sentences, together with the syntax and perhaps semantics of *L*, constitutes a logic of preference in which answers to such questions may be systematically investigated and within which many interesting kinds of preference rankings may be defined. In this chapter we are concerned with formulating a logic of preference of this kind.

In §A a theory of *time-flow*, which will be presupposed throughout, is formulated. This theory is close to that of *Woodger* but with some important differences. In §B, preference is characterized in a preliminary way as a four-place relation involving a person, two distinct sentences of L, and a time. Several different kinds of *preference rankings* are then introduced. The discussion is made more systematic in §C where the logical character of the pragmatical metalanguage is characterized more carefully and suitable *General Rules of Preference* are formulated. A relation of *indifference* is introduced in §D as well as the notion of a *rational preference ranking*. Some further types of preference rankings are introduced in §E, *normal* preference rankings for *tilde*, for *vee*, and the like. In §F, an alternative relation of preference as explicitly based on given *evidence* is considered. Finally, in §G, a few remarks are offered concerning preference between non-linguistic objects.

A. THEORY OF TIME

In his *The Axiomatic Method in Biology* and *The Technique of Theory Construction*[1], Woodger has developed the essentials of a theory of time needed for certain branches of biology. All that appears needed there is a theory of time-flow in which stretches of time are analyzable into shorter stretches and ultimately into *momentary* ones, these momentary ones being not further divisible. No metric or measure for time is introduced, and thus no association between time-stretches and real numbers.

Suppose for the moment that 'x' and 'y' are variables ranging over biological objects of some kind or other. '$x \mathrm{T} y$' then may express that the object x is *before* y in time. This may mean, Woodger explains, that (i) every part of x is temporally before y, or that (ii) x ends at the very moment at which y begins. In this latter case, even though the end of x and the beginning of y temporarily coincide, all the other parts

[1] J. H. Woodger, *The Axiomatic Method in Biology* (Cambridge: Cambridge University Press, 1937), p. 56 ff., and *The Technique of Theory Construction* (*International Encyclopedia of Unified Science*, Vol. II, No. 5, Chicago: University of Chicago Press, 1939), pp. 32–33.

of x if any temporally precede the parts of y. Actually we may let 'x T y' obtain in either of these cases.

Note that for Woodger 'T' stands not for a relation between time-stretches regarded as a new kind of object, but merely for a relation between biological objects already available. Woodger's is thus in a sense a *relative* theory of time, relative to the objects under consideration in the given biological theory.

The theory of time which will be needed for subsequent purposes is in essentials that of Woodger. But here a new kind of variable ranging over time-stretches or intervals, regarded as a new kind of object, is explicitly introduced. Let these be 't', 't_1', 't_2', etc. The intervals may be long or short, the shortest being called *moments*. In place of a relation such as Woodger's T, let us take as primitive a relation of being *wholly before* with its relata as time-stretches. Let 'B' stand for this relation. 't_1 B t_2' is then to be significant and is to read 'the time-stretch t_1 is wholly before the time-stretch t_2'.[2] Note that 'B' incorporates, allowing for the different ontology, only the condition (i) in the foregoing explanation of 'T', i.e., that every part of t_1 wholly precedes every part of t_2.

A time-stretch t_1 may be said to *overlap* with a time-stretch t_2 provided neither stands in B to the other. Thus we may let

$$\text{`}t_1 \mathrm{~O~} t_2\text{'} \quad \text{abbreviate} \quad \text{`}({\sim} t_1 \mathrm{~B~} t_2 \,.\, {\sim} t_2 \mathrm{~B~} t_1)\text{'}.$$

And a time-stretch t_1 is a *part of* a time-stretch t_2 if every time-stretch which overlaps with t_1 overlaps also with t_2.

$$\text{`}t_1 \mathrm{~P~} t_2\text{'} \quad \text{abbreviates} \quad \text{`}(t)(t \mathrm{~O~} t_1 \supset t \mathrm{~O~} t_2)\text{'}.$$

Finally, a *moment* or momentary time-stretch is a part of all its parts.

$$\text{`Mom } t\text{'} \quad \text{abbreviates} \quad \text{`}(t_1)(t_1 \mathrm{~P~} t \supset t \mathrm{~P~} t_1)\text{'}.$$

As *axioms* concerning B, we assume the following.

BR1. $\vdash {\sim} t \mathrm{~B~} t$.

BR2. $\vdash (t \mathrm{~B~} t_1 \,.\, t_1 \mathrm{~B~} t_2) \supset t \mathrm{~B~} t_2$.

BR3. $\vdash (\mathrm{E}t){\sim}\mathrm{Mom}\ t$.

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[2] The treatment of time here is essentially that of *Toward a Systematic Pragmatics*, pp. 36–37.

BR4. $\vdash \sim \mathrm{Mom}\ t \supset (\mathrm{E}t_1)(\mathrm{E}t_2)(\mathrm{Mom}\ t_1 \,.\, \mathrm{Mom}\ t_2 \,.\, t_1\ \mathrm{B}\ t_2 \,.\, t_1\ \mathrm{P}\ t \,.\, \sim(\mathrm{E}t_3)(\mathrm{Mom}\ t_3 \,.\, t_3\ \mathrm{P}\ t \,.\, t_3\ \mathrm{B}\ t_1) \,.\, t_2\ \mathrm{P}\ t \,.\, \sim(\mathrm{E}t_3)(\mathrm{Mom}\ t_3 \,.\, t_3\ \mathrm{P}\ t \,.\, t_2\ \mathrm{B}\ t_3) \,.\, (t_3)((t_1\ \mathrm{B}\ t_3 \,.\, t_3\ \mathrm{B}\ t_2) \supset t_3\ \mathrm{P}\ t))$.

BR5. $\vdash t_1\ \mathrm{P}\ t_2 \supset ((t_2\ \mathrm{B}\ t \supset t_1\ \mathrm{B}\ t) \,.\, (t\ \mathrm{B}\ t_2 \supset t\ \mathrm{B}\ t_1))$.

BR6. $\vdash t_1\ \mathrm{O}\ t_2 \supset (\mathrm{E}t)(t\ \mathrm{P}\ t_1 \,.\, t\ \mathrm{P}\ t_2)$.

BR1 and *BR2* lay down obvious properties of B, namely, that it is an irreflexive and transitive relation. *BR3* is an existence law stipulating that there is at least one nonmomentary time-stretch. For present purposes, this is all that need be assumed directly by way of existence. *BR4* stipulates that any nonmomentary time-stretch contains two momentary parts t_1 and t_2, t_1 being wholly before t_2, such that no momentary part of t is wholly before t_1, and t_2 is before no momentary part of t, and every time-stretch t_3 between t_1 and t_2, so to speak, is also a part of t. *BR4* is a kind of principle of *continuity*, because it assures that nonmomentary time-stretches contain no gaps. From *BR4* and *BR3* it follows that there exist at least two distinct moments. *BR5* stipulates that if one time-stretch t_1 is a part of another t_2, then any time-stretch to which t_2 bears B, t_1 does also, and any time-stretch which bears B to t_2 bears B also to t_1. Finally, *BR6* stipulates that overlapping time-stretches have a common part.

These axioms are not necessarily independent of each other, nor are they intended to provide a complete theory of time-flow. They merely provide the properties which will be needed in the sequel. For other purposes a much more carefully developed theory of time might be needed. And if a metric should be required this may easily be supplied.

We suppose 'B', then, added as an additional primitive to the semantical metalanguage of (I, C), together with the new variables for time-intervals. And the rules *BR1*–*BR6* are to be added to the preceding syntactical and semantical rules.

Identity as a relation between time-stretches may be defined in terms of 'P' as follows.

'$t_1 = t_2$' abbreviates '$(t_1\ \mathrm{P}\ t_2 \,.\, t_2\ \mathrm{P}\ t_1)$'.

The relative consistency of *BR1*–*BR6* may easily be shown by constructing a "model" or interpretation for them within the preceding

syntax. This may be seen as follows. As the range of the variables 't', etc., we take the class whose only members are ex, ac, and $(ex \cap ac)$. We may then define

$$`t_1 \mathrm{B}' t_2\text{' as }`(t_1 = ex \,.\, t_2 = ac)\text{'}.$$

't_1 O$'$ t_2', 't_1 P$'$ t_2', and 'Mom$'$ t' are then defined just as the analogous notions are defined above.

We have then that ex O$'$ ex, ac O$'$ ac, ex O$'$ $(ex \cap ac)$, ac O$'$ $(ex \cap ac)$, and $(ex \cap ac)$ O$'$ $(ex \cap ac)$, $(ex \cap ac)$ O$'$ ex, and $(ex \cap ac)$ O$'$ ac. Also ex P$'$ ex, ac P$'$ ac, $(ex \cap ac)$ P$'$ $(ex \cap ac)$, ex P$'$ $(ex \cap ac)$, and ac P$'$ $(ex \cap ac)$. And ex and ac are the only Mom$'$'s, $(ex \cap ac)$ being the only non-Mom$'$.

On these new meanings for 'B', 'O', etc., we notice that *BR1–BR6* now hold. More specifically, if in place of 'B', 'Mom', 'O', and 'P' in *BR1–BR6* we put in now respectively 'B$'$', 'Mom$'$', 'O$'$', and 'P$'$', we notice that the formulae which result are provable in the underlying syntax. Hence any inconsistency to be gained from *BR1–BR6* may be transformed into an inconsistency within the underlying syntax.

A few elementary theorems concerning time, which may be needed later, are given as follows.

TA1. $\vdash t_1 \mathrm{B}\, t_2 \supset \sim t_2 \mathrm{B}\, t_1$.

By *BR1* and *BR2*.

TA2. $\vdash t_1 = t_2 \vee t_1 \mathrm{B}\, t_2 \vee t_2 \mathrm{B}\, t_1 \vee t_1 \mathrm{O}\, t_2$.

TA3. $\vdash t_1 \mathrm{O}\, t_2 \supset t_2 \mathrm{O}\, t_1$.

TA4. $\vdash t \mathrm{O}\, t$.

TA5. $\vdash t \mathrm{P}\, t$.

TA6. $\vdash (t_1 \mathrm{P}\, t_2 \,.\, t_2 \mathrm{P}\, t_3) \supset t_1 \mathrm{P}\, t_3$.

TA7. $\vdash (\mathrm{E}t)\mathrm{Mom}\, t$.

By *BR3* and *BR4*.

TA8. $\vdash (t)(\mathrm{E}t_1)(\mathrm{Mom}\, t_1 \,.\, t_1 \mathrm{P}\, t)$.

By *TA5*, *TA7*, and *BR4*.

TA9. $\vdash (\mathrm{Mom}\, t_1 \,.\, \mathrm{Mom}\, t_2 \,.\, t_1 \mathrm{P}\, t_2) \supset t_1 = t_2$.

TA10a. $\vdash t_1 \mathrm{B}\, t_2 \supset \sim t_1 \mathrm{O}\, t_2.$

TA10b. $\vdash t_1 \mathrm{P}\, t_2 \supset t_1 \mathrm{O}\, t_2.$

TA10c. $\vdash t_1 \mathrm{B}\, t_2 \supset \sim t_1 \mathrm{P}\, t_2.$

TA11. $\vdash (\mathrm{E}t)(t \mathrm{P}\, t_1 \,.\, t \mathrm{P}\, t_2) \supset t_1 \mathrm{O}\, t_2.$

TA12a. $\vdash (t_1 = t_2 \,.\, t_1 \mathrm{B}\, t_3) \supset t_2 \mathrm{B}\, t_3.$

TA12b. $\vdash (t_1 = t_2 \,.\, t_3 \mathrm{B}\, t_1) \supset t_3 \mathrm{B}\, t_2.$

TA12c. $\vdash (t_1 = t_2 \,.\, t_1 \mathrm{O}\, t_3) \supset t_2 \mathrm{O}\, t_3.$

TA12d. $\vdash (t_1 = t_2 \,.\, \mathrm{Mom}\, t_1) \supset \mathrm{Mom}\, t_2.$

TA13. $\vdash (\mathrm{Mom}\, t_1 \,.\, t_1 \mathrm{O}\, t_2) \supset t_1 \mathrm{P}\, t_2.$

TA14. $\vdash (\mathrm{Mom}\, t_1 \,.\, \mathrm{Mom}\, t_2 \,.\, \sim t_1 = t_2) \supset (t_1 \mathrm{B}\, t_2 \vee t_2 \mathrm{B}\, t_1).$

By *TA13* and *TA2*.

B. PREFERENCE AND CHOICE

We now extend further the semantical metalanguage discussed in Chapter I and the preceding section to take account in certain ways of the human *users* of the language under investigation. In particular we shall consider an experimental situation in which an experimenter E is interested in establishing whether certain relations hold between a human person X and a sentence or sentences of a language at a given time t, and in investigating systematically how such relations are interconnected. It is assumed that E himself knows the language, its syntactical and semantical rules, and its inductive logic, and that he conducts his observations and experiments concerned with X under controlled laboratory conditions. A detailed description of these conditions need not be given. But the logic, deductive and inductive, which E himself uses in his investigation, i.e., in the metalanguage, has been outlined in the preceding chapter.

Suppose E presents the subject X with two sentences a and b of the language L at time t. And suppose he *asks* X which of the sentences X regards as the *more probable*, or in which he *more firmly believes*, or whether he is more willing to *accept* one than the other, or whether he

is more willing to *base his actions* on one than on the other. E then himself accepts and perhaps records a sentence of the metalanguage to the effect that X at time t indicates preference for sentence a over sentence b, if such is the case. Strictly E gives X the *choice* of the two sentences and records which of the sentences X has chosen. Several hours later, or the next day or month, E might repeat the experiment on X and get the same choice in response. E would reasonably infer, there being no evidence to the contrary, that X *would have made* the same choice at any time in between. In other words, E concludes that X *prefers* a to b during the whole time interval. And conversely, if E knows that X prefers a to b during some time interval, he is then presumably willing to infer that X would choose a to b at any time during that interval. Preference then may perhaps be handled as continuous choice (whether explicitly verbalized and recorded or not) throughout some time-interval. If this way of distinguishing between choice and preference is sound, there will be no need of two primitives to handle these notions. The difference can be handled merely in terms of the accompanying theory of time.

Perhaps E would wish to establish that X prefers a to b at t in some other way, in particular, by observing whether X *behaves* or *acts* as though he regarded a as more likely than b at t. If so, E would devise an experiment or series of experiments so that in principle given any two sentences he could decide the one in accord with which X would or does actually behave.

In experimental practice, E might use both of these methods, letting the results of one verify and corroborate the results of the other. Rarely are observations made in isolation. Rather a whole cluster of them are made almost simultaneously and on the basis of a conceptual background. And an experiment involves not only the cluster of observations and the conceptual background, but usually various operations or actions on the part of the experimenter as well. To give a full description of the whole experiment E would make, in order to accept a sentence stating that X prefers a to b at t, would thus involve a good deal and need not be given. For the purposes of what follows, in fact, we should not wish to settle once and for all the exact character of the kind of experiment being considered. For some purposes one kind would be suitable, for others, another. Leaving open the matter

of detail, we shall have greater freedom in interpreting the theory in different ways and for different purposes as needed.

Let 'Prfr' be a new primitive of the metalanguage, and let it stand for a quadratic relation among a person X, sentences a and b, and a time t. Thus let

(1) 'X Prfr a, b, t'

be significant and be read 'X prefers a to b at t'. From what has just been said, it is clear that the precise interpretation of 'Prfr' is left open and we shall be free to interpret it in various ways.

A notion of preference has frequently been discussed by philosophers, especially moral philosophers, as well as by economists and social scientists generally. The basic linguistic form for statements of preference, with a time factor made explicit, seems to be essentially (1), but where 'a' and 'b' are taken as ranging over pleasures or pains, or over concrete goods or economic commodities or the like. The proposal here is that we construe preference only as between sentences, as has already been noted. But there will be many similarities between preference as a relation between sentences and preference as a relation between goods or commodities or pleasures or pains. Thus, for example, it might be reasonable to assume that if X prefers a to b at t and b to c at t he then also prefers a to c at t. Such an assumption, and similar ones, have frequently been made by moral philosophers and economists. We shall *refrain* from making them here, although they may well obtain in many cases. The effect of such assumptions can be achieved in another way.

Suppose X is known or found by the experimenter E to prefer the works of some painters to those of others, at least with respect to certain characteristics.[3] Suppose, e.g., that X is known or found by E to prefer Titian to Orozco as a painter of nudes. E may state this preference by saying something about X's preferences between sentences. Let L be such as to contain certain sentences concerning painters. If X prefers the sentence 'Titian is a fine painter of nudes' of L to the sentence 'Orozco is a fine painter of nudes', he then presumably prefers Titian to Orozco as a fine painter of nudes, and conversely.

[3] This example is an adaptation of one due to Davidson, McKinsey, and Suppes, "Outlines of a Formal Theory of Value, I."

(We may suppose that 'fine' here is understood in some definite way.) Preference between sentences may thus perhaps be regarded as the basic notion, as has already been suggested, and preferences between goods can then be introduced by definition. (See §G below.)

Let the sentences 'Titian is a fine painter of nudes' and 'Orozco is a fine painter of nudes' be sentences of L abbreviated respectively by 'a_t' and a_o'. Let 'a_d' abbreviate 'Delacroix is a fine painter of nudes', another sentence of L. By observation or interrogation E might find that

$$X \text{ Prfr } a_t, a_o, t \,.\, X \text{ Prfr } a_o, a_d, t \,.\, X \text{ Prfr } a_t, a_d, t,$$

or that

$$X \text{ Prfr } a_o, a_d, t \,.\, X \text{ Prfr } a_d, a_t, t \,.\, X \text{ Prfr } a_o, a_t, t,$$

$$X \text{ Prfr } a_o, a_t, t \,.\, X \text{ Prfr } a_t, a_d, t \,.\, X \text{ Prfr } a_o, a_d, t,$$

$$X \text{ Prfr } a_d, a_t, t \,.\, X \text{ Prfr } a_t, a_o, t \,.\, X \text{ Prfr } a_d, a_o, t,$$

$$X \text{ Prfr } a_d, a_o, t \,.\, X \text{ Prfr } a_o, a_t, t \,.\, X \text{ Prfr } a_d, a_t, t,$$

or

$$X \text{ Prfr } a_t, a_d, t \,.\, X \text{ Prfr } a_d, a_o, t \,.\, X \text{ Prfr } a_t, a_o, t.$$

In fact there are just these six possibilities. Each one of these gives a ranking of a_t, a_d, and a_o, a *rational hegemonic ranking* as it is sometimes called. Each one is compatible with the assumption that Prfr is transitive and, further, asymmetric in the sense that for all X, a, b, and t, if X Prfr a, b, t then it is not the case that X Prfr b, a, t.[4]

As another example, let 'a_t' abbreviate the sentence of L 'Tea is a stimulating drink', 'a_o' the sentence 'Ovaltine is a stimulating drink', and 'a_d' 'Drambuie is a stimulating drink'. We may then form rational hegemonic rankings as above for these three sentences, each such ranking describing X's preferences at the time as regards these three beverages with respect to the property of being stimulating.

Within a rational hegemonic ranking we notice that each item is ranked either above or below, so to speak, each other item. If we admit a relation of *indifference* as between a and b, or equality of preferential status, still further types of rankings may be introduced. Let 'Indiff'

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[4] We shall say that Prfr is *transitive* in the sense that if X Prfr a, b, t and X Prfr b, c, t then also X Prfr a, c, t for all X, a, b, c, and t. Strictly this is merely a certain kind of quasi- or partial transitivity for quadratic relations. Note that the use just made in the text of 'asymmetric' is analogous.

stand for the relation of indifference. (The precise definition will be given in §E below.) The experimenter E might then find that the following ranking of preference obtains:

$$X \text{ Prfr } a_o, a_t, t \,.\, X \text{ Indiff } a_t, a_d, t \,.\, X \text{ Indiff } a_d, a_t, t \,.\, X \text{ Prfr } a_o, a_d, t.$$

Such a ranking is compatible with the assumption that Indiff is transitive, and also symmetric in the sense that for all X, a, b, t, if X Indiff a, b, t then X Indiff b, a, t.

We might also expect to find the following kind of ranking:

$$X \text{ Indiff } a_d, a_t, t \,.\, X \text{ Indiff } a_t, a_d, t \,.\, X \text{ Prfr } a_d, a_t, t \,.\, X \text{ Prfr } a_t, a_o, t \,.\, X \text{ Prfr } a_d, a_o, t.$$

But if X prefers a_d to a_t, he is then presumably not indifferent between them, so that this kind of ranking would never be met with.

Another assumption concerning preference and indifference which might be made is that, given any two sentences a and b under consideration at the time t, X prefers a to b at t or b to a at t or is indifferent as between them. If we should add this assumption to the requirements that preference be transitive and asymmetric and that indifference be transitive, we should have a rational hegemonic ranking of the sentences under consideration in which every sentence is assigned a rank in relation to every other sentence, either above or below, so to speak, or at the same rank. Rankings of this kind are especially important and are sometimes called *rational preference rankings*.

These rough introductory remarks concerning rankings are intended merely to indicate some of the interesting kinds of rankings of preference we shall be able to define. There will be other kinds also, concerned with the logical constants *vee*, *tilde*, and so on. X's preferences may or may not exhibit a ranking in accord with the logically correct use of such constants, as we shall see.

Suppose X is an expert in the inductive logic of L and that, given any sentence a of L, he can compute correctly and quickly the degree of confirmation of a on suitable evidence. For the purposes of the theory of preference we are considering, such a person may be regarded as an ideal subject. Given any two sentences of L, suppose that he immediately computes their degrees of confirmation on suitable evidence, compares them, and prefers that with the higher to that with the lower

or, if they have the same degrees, is indifferent as between them. If X's preferences are always of this kind, they are securely founded upon the theory of confirmation or inductive logic of L. His preferences may be taken as paradigmatic; they are never in error and always reflect the correct "probabilities" involved. Such a person we shall refer to occasionally in the sequel. Of course such an "ideal type" might not exist. But presumably a computer or machine could be constructed which would behave as such an ideal type for sufficiently simple L. And it may frequently be useful to have available such a paradigm for purposes of comparison.

C. GENERAL RULES OF PREFERENCE

Let us consider in more detail now the precise logical structure of the metalanguage which arises from that of §A above by adding 'Prfr' as a new primitive.

A new set of variables, 'X', 'Y', 'Z', with or without accents or numerical subscripts, is needed, ranging over an appropriate class of human beings or other users of the language.

We have indicated above some of the conditions concerning Prfr which we should *not* wish to postulate in general. Clearly the experimenter E will not wish to impose restrictions of any kind upon X's preferences, as we have already noted. He will wish rather to await the results of experiment before stating what X's preferences are or that X's preferences exhibit such and such a ranking. Thus any general axioms or rules to be laid down should be extremely weak.

For the present, in fact, only two general rules governing 'Prfr' are needed. The first states that if X prefers a to b at t, then a and b are distinct sentences of L.

PR1. $\vdash X \text{ Prfr } a, b, t \supset (\text{Sent } a \,.\, \text{Sent } b \,.\, {\sim} a = b)$.

The experimental situation is such that only sentences of the language are being tested, so to speak, and in any single experiment just two of them.

Also we assume that if X prefers a to b at t he then prefers a to b at every *momentary part* of t and conversely.

PR2. $\vdash X \text{ Prfr } a, b, t \equiv (t_1)((\text{Mom } t_1 \,.\, t_1 \text{ P } t) \supset X \text{ Prfr } a, b, t_1)$.

This rule makes explicit the relationship between choice and preference as discussed above, choice being regarded merely as momentary preference.

Note that these rules are general in the sense that they hold in effect for all a, b, X, and t. Hence we may refer to them as *General Preference Rules*. Further rules, describing the preferences of particular persons for particular sentences at particular times, may be added as required. For this we should need of course additional constants designating specific users and particular times, structural descriptions being available for sentences. Such rules may be called *Specific Preference Rules*. No matter how the primitive 'Prfr' is interpreted, we should have presumably rules of both kinds.

Note that the consistency of the two General Rules relative to the underlying syntax or semantics (plus theory of time) may be proved by letting 'X Prfr a, b, t' mean either

$$X = X \;.\; \mathrm{Sent}\, a \;.\; \mathrm{Sent}\, b \;.\; {\sim} a = b \;.\; t = t$$

or

$$X = X \;.\; \mathrm{Tr}\, a \;.\; \mathrm{Tr}\, b \;.\; {\sim} a = b \;.\; t = t.$$

On either of these interpretations, *PR1* and *PR2* are provable (using *TA8* above).

With the new primitive 'Prfr', further rules of identity are needed for the variables and constants for time and for human beings, analogous to *IdR1-IdR3* of (I, B). Note however that in view of *PR2* we have that

$$\vdash (\mathrm{Mom}\, t_1 \;.\; t_1 = t_2) \supset (X \;\mathrm{Prfr}\; a, b, t_1 \equiv X \;\mathrm{Prfr}\; a, b, t_2).$$

Hence here we need assume merely that

$$\vdash ({\sim}\mathrm{Mom}\, t_1 \;.\; t_1 = t_2) \supset (X \;\mathrm{Prfr}\; a, b, t_1 \equiv X \;\mathrm{Prfr}\; a, b, t_2).$$

D. INDIFFERENCE AND RATIONAL PREFERENCE RANKINGS

It terms of 'Prfr' the relation of indifference may be defined, as has already been suggested. If X neither prefers a to b at some momentary time t_1 nor b to a at t_1, X then is presumably indifferent as between a and b at t_1. This notion we may extend to nonmomentary times t by

requiring that X be indifferent as between a and b at every momentary part of t. Thus

'X Indiff a, b, t' may abbreviate '(Sent a . Sent b . (t_1) ((Mom t_1 . t_1 P t) $\supset$ ($\sim X$ Prfr a, b, t_1 . $\sim X$ Prfr b, a, t_1)))'.

Concerning this relation, we can prove rules analogous to *PR1* and *PR2*.

TD1. $\vdash X$ Indiff $a, b, t \supset$ (Sent a . Sent b).

TD2. $\vdash$ (Sent a . Sent b) $\supset$ (X Indiff $a, b, t \equiv (t_1)$ ((Mom t_1 . t_1 P t) $\supset$ X Indiff a, b, t_1)).

The proof utilizes *TA8* and *TA9*.

Also we note that

TD3. $\vdash X$ Indiff $a, b, t \supset$ ($\sim X$ Prfr a, b, t . $\sim X$ Prfr b, a, t).

The proof utilizes *TA8* and *PR2*.

TD4. $\vdash X$ Indiff $a, b, t \supset X$ Indiff b, a, t.

TD5. $\vdash$ Sent $a \supset X$ Indiff a, a, t.

Note that the transitivity (more precisely, a quasi-transitivity) of Prfr has not been postulated, i.e., we do not postulate that if X prefers a to b at t, and b to c at t, he then also prefers a to c at t, for all X, a, b, c, and t. Of course this circumstance may obtain in special cases. Also transitivity (more precisely, quasi-transitivity) for indifference is not assumed, but may obtain in special cases. Nor is it postulated that either X prefers a to b at t, or b to a at t, or else that he is indifferent as between them at t. Of course this circumstance also may obtain for certain X and for certain virtual classes of sentences, but we do not wish to postulate it in general. We might also require that one and only one of these three conditions hold for any X at t. But this again we should not wish to postulate in general.

The four conditions mentioned are often taken by economists, and social scientists generally, as constituting conditions of *rational* behavior. If X's preferences are in accord with these requirements, his preferences are in effect said to be *rational*. Using this customary

terminology, we may introduce the notion of a *rational preference ranking* (RPR).

In view of *TD3*, however, the actual definition of a rational preference ranking may be given without the fourth condition mentioned, i.e., that one and only one of the three circumstances (that X Prfr a, b, t, X Prfr b, a, t, or X Indiff a, b, t) obtains. But in place of this fourth condition we should have that if X Prfr a, b, t then it is not the case that X Prfr b, a, t.

More specifically, we may say that a virtual class of sentences containing at least three distinct sentences exhibits a rational preference ranking relative to X and t as follows.

'F RPR X, t' abbreviates '$((\mathrm{E}a)(\mathrm{E}b)(\mathrm{E}c)(F a \,.\, F b \,.\, F c \,.\, {\sim} a = b \,.\, {\sim} a = c \,.\, {\sim} b = c) \,.\, (a)(b)(c)((F a \,.\, F b \,.\, F c \,.\, X \text{ Prfr } a, b, t \,.\, X \text{ Prfr } b, c, t \,.\, {\sim} a = c) \supset X \text{ Prfr } a, c, t) \,.\, (a)(b)(c)((F a \,.\, F b \,.\, F c \,.\, X \text{ Indiff } a, b, t \,.\, X \text{ Indiff } b, c, t) \supset X \text{ Indiff } a, c, t) \,.\, (a)(b)((F a \,.\, F b) \supset (X \text{ Prfr } a, b, t \vee X \text{ Prfr } b, a, t \vee X \text{ Indiff } a, b, t)) \,.\, (a)(b)((F a \,.\, F b \,.\, X \text{ Prfr } a, b, t) \supset {\sim} X \text{ Prfr } b, a, t))$'.

The clause requiring that the virtual class considered must contain at least three distinct sentences, in the definiens of this definition, is perhaps not strictly needed. But without it, the null virtual class, as well as any virtual class of expressions not containing at least three distinct sentences, would exhibit an RPR with respect to all X and all t, and this quite irrespective of X's actual preferences. To allow this might seem counterintuitive in some sense. Similar existence provisos will be put upon F in many of the subsequent definitions.

In view of *TD3*, we have obviously the following theorem.

TD6. $\vdash (F \text{ RPR } X, t \,.\, F a \,.\, F b) \supset ({\sim}(X \text{ Prfr } a, b, t \,.\, X \text{ Prfr } b, a, t) \,.\, {\sim}(X \text{ Prfr } a, b, t \,.\, X \text{ Indiff } a, b, t) \,.\, {\sim}(X \text{ Prfr } b, a, t \,.\, X \text{ Indiff } a, b, t))$.

TD7 stipulates a circumstance under which the Boolean product of two virtual classes exhibits an RPR.

TD7. $\vdash ((F \text{ RPR } X, t \vee G \text{ RPR } X, t) \,.\, (\mathrm{E}a)(\mathrm{E}b)(\mathrm{E}c)((F \cap G)\, a \,.\, (F \cap G)\, b \,.\, (F \cap G)\, c \,.\, {\sim} a = b \,.\, {\sim} b = c \,.\, {\sim} a = c)) \supset (F \cap G) \text{ RPR } X, t$.

TD8. $\vdash ((F \cup G)$ RPR $X, t \,.\, (\mathrm{E}a)(\mathrm{E}b)(\mathrm{E}c)(F\,a \,.\, F\,b \,.\, F\,c \,.\, \sim a = b$ $\sim b = c \,.\, \sim a = c)) \supset F$ RPR X, t.

We need not draw out consequences in laborious detail of the various definitions. But occasionally a few theorems will be given, as here, some obvious, some less so, enunciating significant properties of the notions introduced.

Strictly, it should be recalled, we are using 'F', 'G', etc., to stand for expressional abstracts of the metalanguage. These letters themselves are therefore notations within the metametalanguage. Frequently we shall find it convenient in what follows to use such phrases as 'the virtual class F', 'F contains at least one member', etc., to mean 'the virtual class for which F stands', 'the virtual class for which F stands contains at least one member', and so on. In the strict sense phrases such as 'the virtual class F' are meaningless, 'F' standing for an abstract not a virtual class. (Properly, for consistency of usage, we should write 'F', etc., in boldface.) Further, the virtual classes involved are to be understood throughout as virtual classes of sentences, unless otherwise indicated.

E. SOME FURTHER PREFERENCE RANKINGS

Let us consider now some further preferential rankings, depending upon the presence of such logical constants as *tilde* or *vee* in the sentences under consideration.

Consider some virtual class F of sentences containing at least two different sentences together with their *negations*. For every distinct a and b, if a, b, (*tilde* a), and (*tilde* b) are in F, then if X prefers a to b at t if and only if he prefers (*tilde* b) to (*tilde* a), we may say that the sentences of F exhibit a *normal preference ranking* for *tilde* relative to X at time t, i.e., X's ranking of preference is in accord roughly with the normal, logically correct use of *tilde*, at least as applied to the (whole) sentences of F.

'F NPR *tilde*, X, t' abbreviates '$((\mathrm{E}a)(\mathrm{E}b)(F\,a \,.\, F\,b \,.\, \sim a = b \,.\, F$ (*tilde* a) . F (*tilde* b)) . $(a)(b)((F\,a \,.\, F\,b \,.\, \sim a = b \,.\, F$ (*tilde* a) . F(*tilde* b)) $\supset$ (X Prfr $a, b, t \equiv X$ Prfr (*tilde* b), (*tilde* a), t)))'.

This definition is of interest in connecting X's preferences of sentences of F with the logically correct use of *tilde*. Consider for a moment the ideal or paradigmatic situation discussed above in §B in which X knows correctly the degree of confirmation on suitable evidence e of every sentence in F and that he prefers a to b at t if and only if $\mathrm{c}(a, e) > \mathrm{c}(b, e)$. Then and only then does $\mathrm{c}((\mathit{tilde}\ b), e) > \mathrm{c}((\mathit{tilde}\ a), e)$, and hence X prefers $(\mathit{tilde}\ b)$ to $(\mathit{tilde}\ a)$ at t if and only if he prefers a to b at t.

It might be objected that the '$\equiv$' in the definiens here is too strong. Weaker types of normal preference rankings for *tilde* may be considered by putting '$\supset$' for '$\equiv$' or by weakening the existence requirements for F.

We have the following theorems concerning NPR and the Boolean notions.

TE1. $\vdash ((F\ \mathrm{NPR}\ \mathit{tilde}, X, t \vee G\ \mathrm{NPR}\ \mathit{tilde}, X, t) \,.\, (\mathrm{E}a)(\mathrm{E}b)((F\cap G)\, a \,.\, (F\cap G)\, b \,.\, {\sim} a = b \,.\, (F\cap G)\,(\mathit{tilde}\ a) \,.\, (F\cap G)\,(\mathit{tilde}\ b))) \supset (F\cap G)\ \mathrm{NPR}\ \mathit{tilde}, X, t$.

TE2. $\vdash ((F\cup G)\ \mathrm{NPR}\ \mathit{tilde}, X, t \,.\, (\mathrm{E}a)(\mathrm{E}b)(F\, a \,.\, F\, b \,.\, {\sim} a = b \,.\, F\,(\mathit{tilde}\ a) \,.\, F\,(\mathit{tilde}\ b))) \supset F\ \mathrm{NPR}\ \mathit{tilde}, X, t$.

TE3. $\vdash {\sim}\Lambda\ \mathrm{NPR}\ \mathit{tilde}, X, t$.

In a somewhat similar way, a virtual class F of sentences may exhibit a *normal preference ranking* for *vee*. Suppose F is such as to contain some sentences a, b, and c, and also $(a\ \mathit{vee}\ b)$, where a and b are both distinct from c. Suppose further that for all such sentences, if X prefers a to c at t or b to c at t, then he prefers $(a\ \mathit{vee}\ b)$ to c, and also that if he prefers c to $(a\ \mathit{vee}\ b)$ at t he then prefers c to a and c to b at t. His preferences then may be said to exhibit a normal preference ranking with respect to *vee* relative to F and t.

'$F\ \mathrm{NPR}\ \mathit{vee}, X, t$' abbreviates '$((\mathrm{E}a)(\mathrm{E}b)(\mathrm{E}c)(F\, a \,.\, F\, b \,.\, F\, c \,.\, {\sim} a = c \,.\, {\sim} b = c \,.\, {\sim}(a\ \mathit{vee}\ b) = c \,.\, F\,(a\ \mathit{vee}\ b)) \,.\, (a)(b)(c)((F\, a \,.\, F\, b \,.\, F\, c \,.\, {\sim} a = c \,.\, {\sim} b = c \,.\, {\sim}(a\ \mathit{vee}\ b) = c \,.\, F\,(a\ \mathit{vee}\ b)) \supset (((X\ \mathrm{Prfr}\ a, c, t \vee X\ \mathrm{Prfr}\ b, c, t) \supset X\ \mathrm{Prfr}\ (a\ \mathit{vee}\ b), c, t) \,.\, (X\ \mathrm{Prfr}\ c, (a\ \mathit{vee}\ b), t \supset (X\ \mathrm{Prfr}\ c, a, t \,.\, X\ \mathrm{Prfr}\ c, b, t)))))$'.

Let us consider again the ideal type of user of L whose preferences are wholly determined by the correct degrees of confirmation on suitable

evidence e. If he prefers a to c at t or b to c at t, then $c(a, e) > (c, e)$ or $c(b, e) > c(c, e)$. But in either case, by *TH7* of Chapter I, $c((a \textit{ vee } b), e) > c(c, e)$, and hence he Prfr ($a$ *vee* b) to c at t. Also if he Prfr c to (a *vee* b) at t, then $c(c, e) > c((a \textit{ vee } b), e)$. But then, by *TH8* (Chapter I), both $c(c, e) > c(a, e)$ and $c(c, e) < c(b, e)$ and hence he both Prfr c to a and c to b at t.

Note that stronger conditions are not needed in this definition. If X Prfr (a *vee* b), c, t (i.e., $c((a \textit{ vee } b), e) > c(c, e)$ in the paradigmatic case), then it should not follow that X Prfr a, c, t or X Prfr b, c, t. Similarly a condition that X Prfr c to a or to b at t does not entail that X Prfr c, (a *vee* b), t. Such requirements could of course be added, but they would destroy the paradigmatic character of the definition in the sense taken.

Concerning this notion of a NPR for *vee* theorems analogous to *TE1–TE3* may be proven.

It might well be the case that X does not use or understand *tilde* or *vee* in their normal sense. X's preferences might still exhibit some rankings such as those described in these definitions, using some other symbol or symbols in place of *tilde* or *vee*. Thus we may say that X's preferences among sentences of F exhibit a normal preference ranking at t *with a as a sign for negation*, as follows.

'F NPR *Neg*, a, X, t' abbreviates '$((\mathrm{E}b)(\mathrm{E}c)(F\,b \,.\, F\,c \,.\, \sim b = c \,.\, F\,(a \frown b) \,.\, F\,(a \frown c)) \,.\, (b)(c)((F\,b \,.\, F\,c \,.\, \sim b = c \,.\, F\,(a \frown b) \,.\, F\,(a \frown c)) \supset (X \text{ Prfr } b, c, t \equiv X \text{ Prfr } (a \frown c), (a \frown b), t)))$'.

A similar definition may be given for disjunction.

It should be noted again that the experimenter E never imposes any of these rankings upon X. He experiments and observes whether X's preferences fit the definitions or not. X is quite free to exhibit whatever preferences he chooses, and the rankings may or may not satisfy certain reasonable requirements of consistency or normalcy or rationality. If they do, these various definitions will help E to classify X's preferences by placing them in the proper rankings. Under certain circumstances they might also be of help to X himself, in helping him to revise his preferences so as to accord with one or another of these rankings.

X's preferences among sentences of F may be said to exhibit a *normal preference ranking with respect to LogThm* at t as follows.

'F NPR *LogThm*, X, t' abbreviates '((Ea)(Eb)($F\,a$. $F\,b$. LogThm a . $\sim$Tr b) . (a)(b)(($F\,a$. $F\,b$. LogThm a . $\sim$Tr b) $\supset$ X Prfr a, b, t))'.

Of course X may or may not know logic. But if he consistently prefers logical theorems of F to false sentences of F, his preferences clearly exhibit a ranking in accord with the normal use of logical theorems, all of which have maximal confirmation on any non-logically false evidence (*TH13* of I).

A slightly different definition, more in accord with the paradigmatic use of confirmation, is gained by writing '(X Prfr a, b, t v X Indiff a, b, t)' here in place of 'X Prfr a, b, t'.

Among the logical axioms or rules an especially important one, in view of the use here of virtual classes, is *Abst*. We might say that F exhibits a *normal preference ranking for Abst* relative to X and t as follows, where 'Abst e' expresses that e is an instance of *Abst*, i.e., one of the formulae stipulated by *Abst* to be a logical axiom.

'F NPR *Abst*, X, t' abbreviates '((Ea)(Eb)(Ec)(Ed)(Sent d . InCon a . Vbl b . Fmla c . Abst ($b \cap inrep \cap c \cap a$ *tripbar* d) . F ($b \cap inrep \cap c \cap a$) . F d) . (a)(b)(c)(d)((Sent d . InCon a . Vbl b . Fmla c . Abst ($b \cap inrep \cap c \cap a$ *tripbar* d) . F ($b \cap inrep \cap c \cap a$) . $F\,d$) $\supset$ (e)((X Prfr ($b \cap inrep \cap c \cap a$), e, t $\equiv$ X Prfr d, e, t) . (X Prfr e, ($b \cap inrep \cap c \cap a$), t $\equiv$ X Prfr e, d, t))))'.

An alternative definition is gained by requiring that X Indiff ($b \cap inrep \cap c \cap a$), d, t, in place of the condition concerning Prfr. The two definitions, however, appear equivalent on the supposition that F RPR X, t.

Also, a virtual class F of sentences might be said to exhibit a *normal preference ranking with respect to Thm*, or *Tr*, relative to X at t, respectively, as follows.

'F NPR *Thm*, X, t' abbreviates '((Ea)(Eb)($F\,a$. $F\,b$. Thm a . $\sim$Tr b) . (a)(b)(($F\,a$. $F\,b$. Thm a . $\sim$Tr b) $\supset$ X Prfr a, b, t))',

and

'F NPR *Tr*, X, t' abbreviates '((Ea)(Eb)($F\,a$. $F\,b$. Tr a . $\sim$Tr b) . (a)(b)(($F\,a$. $F\,b$. Tr a . $\sim$Tr b) $\supset$ X Prfr a, b, t))'.

Note that the second conjuncts in the definientia of these definitions give only sufficient, not necessary, conditions for preference. In the case of the definition involving *Tr*, for example, we surely would wish to allow X to prefer a to b without requiring that a be Tr and b not a Tr. To require this latter would be much too strong.

In the definition involving *Thm*, we require that X prefer theorems to falsehoods. It might seem reasonable to require that X prefer theorems to nontheorems. But F might contain an *undecidable* sentence of L, i.e., a sentence which is not a theorem but true. If X preferred a theorem to such a sentence, his preference would not presumably be regarded as normal with respect to *Thm*, in one sense at least of the term 'normal'.

Of course, the very meaning of 'normal' in these contexts is not too clear. For most of them, 'normal' has meant roughly *in accord with the degrees of confirmation* (on suitable evidence e) *involved*, so that X Prfr a, b, t if and only if $\mathrm{c}(a, e) > \mathrm{c}(b, e)$. For some of the definitions, however, 'normal' is used in a different sense. X may well prefer a truth a of F to a nontruth b of F where it is not the case that $\mathrm{c}(a, e) > \mathrm{c}(b, e)$, the confirmation of a sentence on suitable e in no way depending upon its factual truth or falsity.

Finally, we make explicit the use of confirmation. Of course we must presuppose for the moment that the metalanguage contains the logic of confirmation, which we have not done heretofore. We may then say that F exhibits a *normal preference ranking with respect to confirmation* on evidence e relative to X and t as follows. (Here the presence of '$\equiv$' in the definiens is desirable.)

'F NPR *Conf*, X, t, e' abbreviates '$((\mathrm{E}a)(\mathrm{E}b)(F a \,.\, F b \,.\, {\sim} a = b) \,.$ $\mathrm{Sent}\, e \,.\, {\sim}\mathrm{LFls}\, e \,.\, (a)(b)((F a \,.\, F b) \supset (\mathrm{c}(a, e) > \mathrm{c}(b, e) \equiv X\, \mathrm{Prfr}\; a, b, t)))$'.

Suppose X is a paradigmatic user of L of the kind mentioned at the end of §B, whose preferences accord with the proper degrees of confirmation involved. For him, presumably, any virtual class F would exhibit an NPR with respect to confirmation at any time.

These definitions are highly tentative, being mere first attempts at characterizing various kinds of normal preference rankings. No doubt they may be improved. Also they are probably of less interest than the *quantitative* patterns which will be given in the next chapter.

F. PREFERENCE ON GIVEN EVIDENCE

We should note that in the foregoing Prfr and Indiff are not directly governed by the evidence *e*. The evidence *e* has been mentioned only when comparison with the paradigmatic user of *L* is needed.

It might be thought desirable to make explicit the use of *e*. Rational preference might be thought to be governed by the "rational" degrees of confirmation on suitable evidence. *X* rationally prefers *a* to *b* at time *t on evidence e* provided he estimates or takes at time *t* the degree of confirmation of *a* on *e* to be greater than that of *b* on *e*. In place of 'Prfr' we might now write 'Prfr_e,' to stand for this relation. We could then rephrase the preceding definitions, with the necessary changes, using 'Prfr_e' in place of 'Prfr'.

It is not clear that this change is really needed, however. 'Prfr' may always be construed as 'Prfr_e' where *e* is the total available relevant evidence at the time, available, i.e., to *X* or *E* or both at *t* and relevant to the sentences of the class *F*. If 'Prfr_e' should be required for some specific purpose, however, we may presuppose that it has been suitably characterized here experimentally, that appropriate pragmatical rules have been given for it, and that all of the notions introduced in this chapter have been reconstrued now in terms of it. Much of what will be said subsequently can be said with or without explicit mention of *e*.

G. PREFERENCE AS BETWEEN NONLINGUISTIC OBJECTS

It has been suggested above that preference between sentences may be the logically fundamental relation of preference and that preference between nonlinguistic entities may be suitably defined in terms of it. Let us reflect upon this suggestion a little more closely for a moment.

First we consider preference for an individual object a as over and against another object b. Suppose a is a certain cup of tea and b a cup of coffee. To say that a person *X* prefers a to b at time *t* may perhaps be regarded as elliptical for saying that person *X* prefers a to b at *t with respect to such and such a property* P. Suppose P is the property of having such and such a flavor and that *X* estimates that a has this flavor but b does not. We then have that

$$X \text{ Prfr 'Pa', 'Pb'}, t$$

in the sense of 'Prfr' appropriate to sentences. But this is to say that X prefers a to b at t with respect to the property P.

Often statements of preference are as between kinds or classes of objects rather than between objects themselves. That X prefers tea to coffee in general would be expressed by such a statement. Statements of this kind can perhaps be achieved by quantifying the time-parameter. Reference to the property with respect to which there is preference seems needed here also. X may prefer tea to coffee in general with respect to such and such a property, e.g., that of being soothing in the afternoon, but prefer coffee to tea with respect to such and such another property, e.g., that of being stimulating in the morning.

Statements of preference as between properties themselves can presumably be handled in terms of statements of preference concerning objects having those properties. Here also reference to the property with respect to which there is preference seems needed. We may say say that X prefers property P to Q with respect to some M at t provided X prefers x to y with respect to M at t for every x having P and every y having Q. As an example we might say that X prefers the color blue to red with respect to the property of being soothing. This is merely to say that X prefers blue objects to red with respect to being soothing. In a similar way statements of preference as between dyadic relations, etc., may perhaps be handled.

These comments are put forward merely as suggestions and with no attempt to work out the definitions in any detail. But enough has been said to lend some support to the contention that preference between sentences is the logically fundamental relation of preference.

III

DEGREE OF ACCEPTANCE

LET US TURN NOW to the problem of defining a notion of *degree of acceptance* for sentences belonging to some virtual class F of sentences of L. Often in the methodology of the sciences, and in analytic philosophy generally, we wish to say that such and such a sentence or set of sentences is *accepted* by such and such a person or social group. But little attempt has been made to refine this notion in such a way as to introduce a notion of degree of acceptance. No systematic and far-reaching theory has been developed until quite recently which provides a basis for giving a satisfactory account of such a notion. In this chapter one possible way of developing such a theory is considered.

The guiding idea to be used here is due essentially to von Neumann and Morgenstern. In their *Theory of Games and Economic Behavior* the foundations of a theory of utility are laid. This theory has attracted wide attention on the part of social scientists and mathematicians. But unfortunately philosophers and philosophical logicians have shown little interest in this subject and seem for the most part to have been unaware of its philosophical relevance. The reason for this lies in part that heretofore attention has been focussed exclusively on the utility of economic goods or commodities or concrete

objects. Here we shall be concerned with the utility of *sentences* of a language-system L, just as we were concerned with preferences as between sentences in the preceding chapter. The application of what is essentially the von Neumann-Morgenstern theory of utility to the sentences of a well-knit language-system will enable us to define many important and interesting notions which have heretofore for the most part resisted precise definition.

In §A a *new primitive* is introduced and first characterized informally. Then some *pragmatical rules* are laid down governing it. In §B the notion of a *rational preference pattern* is defined. The fundamental *theorem of adequacy* is presented in §C and the exact definition of *degree of acceptance* is then given in §D. In §E several kinds of *acceptance patterns* are presented, some of them rather similar to some of the rankings of preference presented in the preceding chapter. In §F some comments concerning the *social group* are put forward. Finally, in §G, some *alternative* ways of providing the material of this chapter are suggested.

A. THE FUNDAMENTAL RELATION

As a basis for introducing a notion of degree of acceptance and for the material of subsequent chapters, a new primitive relation is required. This is symbolized by 'Eq' which is to occur in contexts of the form

(1) 'X Eq a, b, c, t, α',

read '*X equates a* with the "*combination*" of sentences b and c at time t to the *degree* α'.

In order to give an experimental or operational meaning to expressions of this form, we assume a situation somewhat as follows. The experimenter E can decide, as in the preceding chapter, whether the person X prefers at t a sentence a to a sentence b. X himself, we recall, may or may not be conscious of this preference, but E must be able, by observing X's behavior or asking X various questions, etc., to determine it for any two sentences he (E) picks out. Given three distinct sentences a, b, and c of L, suppose E discovers that X does not prefer a to b or c to a. To state the results of this E needs only the notion 'Prfr' discussed above. What E now wishes is to be able to discuss X's

preference, or rather his indifference, not only as between sentences, but as between sentences and "combinations" or "alternatives" of sentences with stated "probabilities" or degrees of confirmation. In addition to 'Prfr', the new primitive 'Eq' is required to enable him to do this. This new primitive, together with the theory characterizing it, provides a very "natural" extension of the theory of preference in the preceding chapter.

The words 'alternative' or 'combination' are used here only informally to help describe the meaning of 'Eq'. Combinations are not recognized as new kinds of entities as values for variables. But where 'X Eq a, b, c, t, α' holds, a is equated in a kind of way with b and c taken together in a kind of way. So it is convenient to refer to b and c here as forming a combination.

We must distinguish two kinds of interpretations for 'Eq', the paradigmatic and the nonparadigmatic. In the paradigmatic interpretation, α is to be regarded as the confirmation of b and $(1 - \alpha)$ as the confirmation of c (on suitable combined evidence e). In other words, the sentences in the combination are such that their degrees of confirmation on the given evidence sum to 1, so that if one has α as its degree of confirmation, the other will have $(1 - \alpha)$. If X equates a paradigmatically with the combination of b and c, X is then indifferent as between a and the combination of b and c, these latter having degrees of confirmation on the given evidence α and $(1 - \alpha)$ respectively.

Perhaps X is an expert in inductive logic, is familiar with the notion of confirmation, and, given any sentence of L, can compute its degree of confirmation quickly and correctly. But this we need not assume. If X is familiar with the concept of degree of confirmation but is not skilled at computing numerical values, E may *tell* him the degrees of confirmation of the alternatives involved. But this we need not assume either. In fact, X may be completely ignorant of the notion of confirmation, relying on E to supply the numerical measure.

For E to decide in a special case whether a given sentence of the form 'X Eq a, b, c, t, α' holds or not, he may devise some suitable experiment or set of operations. He may *ask* X whether he, X, equates a with the combination of b and c (where X himself is presumed to understand the meaning of 'Eq'). E may observe X's *behavior*, providing it is relevant, to decide. He may do both of these, as well as use

other data about X's behavior, including any relevant preferences or indifferences previously established. In many such ways, perhaps all of them together, E should himself be able to decide which sentences of the form 'X Eq a, b, c, t, α' to accept and which to reject.

Another way for E to decide whether a given sentence of the form 'X Eq a, b, c, t, α' holds paradigmatically is for him to offer X at time t a suitable *bet*. We need not attempt to characterize the precise nature of such a bet. Betting behavior has often been taken as providing an approach to rational decision and may presumably be so taken here.

Instead of seeking an operational meaning for 'Eq', perhaps we should regard it rather as a "theoretical construct," in the sense of Carnap and Hempel.[1] In this case we should not seek to characterize it directly in experimental terms. This is a matter that may be left open for the present.

The notion of confirmation is used here paradigmatically to provide the necessary numerical foothold. Alternatively, where X does not prefer a to b at t, nor c to a, the measure α may be taken as *the numerical ratio for his preference of a over c to that of b over c.* This may be seen as follows.

Scientists in their official parlance provide perhaps ideal subjects for us to consider. Scientists have a proclivity for sentences well-confirmed on the relevant available evidence rather than for sentences with only low confirmation on that evidence, and their preferences between sentences should reflect this proclivity to some extent. At any event it is not unreasonable in the present context to assume this. The higher the degree of confirmation of a sentence b on the available evidence the greater is the scientist's "satisfaction" with b, so to speak, at least for sentences b in his special discipline. Of course not all of scientists' discourse is scientific. Nonetheless this example is clearly useful as an ideal case.

Let us speak informally, for the moment, of the satisfaction to be gained from b as the "utility" of b, and let it be designated by '$\mathrm{u}(b)$'. If person or scientist X equates a with the combination of b and c,

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[1] Cf. Hempel, *op. cit.* Also R. Carnap, "The Methodological Status of Theoretical Concepts," in *Minnesota Studies in the Philosophy of Science*, Vol. I (Minneapolis: University of Minnesota Press, 1956), pp. 58–76.

then his satisfaction with a, u(a), should equal roughly the sum of (i) his satisfaction with b, u(b), times its likelihood of being true (degree of confirmation) with (ii) his satisfaction from c, u(c), times its likelihood of being true, i.e., his utility from a should equal the likely utility from the combination of b with c. This circumstance is described by the following "natural" kind of equation, where α is the confirmation of b, $(1 - \alpha)$ of c, on suitable e.

$$(2) \qquad \mathrm{u}(a) = ((\alpha \,.\, \mathrm{u}(b)) + ((1 - \alpha) \,.\, \mathrm{u}(c))).$$

Solving for α, provided $(\mathrm{u}(b) - \mathrm{u}(c)) \neq 0$, we have that

$$(3) \qquad \alpha = ((\mathrm{u}(a) - \mathrm{u}(c))/(\mathrm{u}(b) - \mathrm{u}(c))).$$

Hence, as suggested, α may be regarded as providing the ratio for X's preference of a over c to that of b over c at the time t.

Note that we have spoken of confirmation here only in the informal explanation. 'Eq' may be so interpreted that the α involved is a suitable degree of confirmation. This is not essential, however, for the present theory. Alternatively α may be taken merely as an arbitrary ratio of preference (in accord with (3)) in which case no specifically "rational" or paradigmatic or objective factor is used. Strictly there are many alternative interpretations for 'Eq', just as for 'Prfr'. Leaving open details, we shall be free to interpret 'Eq' and 'Prfr' in various ways as required in specific contexts.

Several alternative ways of construing α in atomic sentences of the form 'X Eq a, b, c, t, α' have then in effect been distinguished. α may be the "correct" degree of confirmation of b on the total evidence available to X or to E or to both at the time. α may be taken not as the correct degree of confirmation but only as the estimate of such, as estimated by X or E or both at the time. Or α may be taken, by X or E or both, as the ratio of X's preference (at the time) of a over c to that of b over c without regard to degrees of confirmation. All of these ways of construing α should be explored and some of them no doubt are more suitable than others for certain tasks or purposes.

The various meanings of 'Eq' are thus seen to be rather complicated. But this apparently cannot be helped. Much important experimental and theoretical work is being done in contemporary decision theory

which will no doubt help to clarify, sharpen, and perhaps simplify these meanings.

The use of real numbers in the way described provides the effect of an *interval measure* of preference. That such a measure gives rise to a system of numerical measurement for utility has frequently been noted in the literature of economics, and was apparently first observed by Vilfredo Pareto in his *Manuel d'Économie Politique*.[2]

We note that E uses 'Eq', howsoever interpreted, in such a way that the following rules are clearly to hold. The first two are analogous to *PR1* and *PR2* and the third merely serves to make explicit one of the informal comments above.

EqR1. $\vdash X \text{ Eq } a, b, c, t, \alpha \supset (\text{Sent } a \,.\, \text{Sent } b \,.\, \text{Sent } c \,.\, {\sim} a = b \,.\, {\sim} a = c \,.\, {\sim} b = c \,.\, 0 \leqq \alpha \leqq 1)$.

EqR2. $\vdash X \text{ Eq } a, b, c, t, \alpha \equiv (t_1)((\text{Mom } t_1 \,.\, t_1 \text{ P } t) \supset X \text{ Eq } a, b, c, t_1, \alpha)$.

EqR3. $\vdash X \text{ Eq } a, b, c, t, \alpha \supset ({\sim} X \text{ Prfr } a, b, t \,.\, {\sim} X \text{ Prfr } c, a, t)$.

Note that these rules, like *PR1* and *PR2*, are in no way *imposed* upon X's preferences. Rather they are rules used by the experimenter E to govern *his* discourse concerning X's preferences. X's preferences may exhibit all manner of inconsistencies, but E's discourse should clearly be in accord with the rules of the quantitative metalanguage.

The consistency of *EqR1–EqR3*, relative to *PR1–PR2* and the underlying quantitative semantics, may be shown by taking

$$`X \text{ Eq } a, b, c, t, \alpha'$$

to mean that

$$X \text{ Indiff } a, b, t \,.\, X \text{ Indiff } a, c, t \,.\, {\sim} a = b \,.\, {\sim} b = c \,.\, {\sim} a = c \,.\, 0 \leqq \alpha \leqq 1.$$

With 'Eq' so construed it is easily seen that *EqR1–EqR3* are provable in the preceding theory.

To *EqR1–EqR3*, E will often wish to add some specific statements describing X's equatings of specific sentences at given times. Such

[2] 2nd ed. (Paris: M. Giard, 1927).

statements may be regarded as additional rules or used as hypotheses wherever needed. Such rules would be *Specific Eq Rules*, *EqR1–EqR3* being *General Eq Rules*.

It is to be remarked that in the extended metalanguage a new style of variables has been admitted taking real numbers as values. The legitimacy of such wholesale admission of real numbers in this way may be questioned both on grounds of ontological economy and from the point of view of mathematical constructivism.[3] Perhaps in fact the rationals alone may be made to suffice. For subsequent purposes, however, we shall need certain theorems (*TC1* and *TC2* below) which seem to require the full real number system for their proof. Perhaps those proofs can be reconstrued so as to involve only the rationals. This is a matter which must be left open for the present.

With 'Eq' adopted as a new primitive, the preceding rules of identity (II, C) must be extended. Also further rules of identity must now be given for the real number variables and constants.

B. RATIONAL PREFERENCE PATTERNS

The notion of a rational preference *ranking* of the sentences of F was introduced in (II, D). In terms of 'Eq' we may now introduce a more sophisticated, pliable, and refined notion of a rational preference *pattern*.

Consider again a large virtual class F of sentences of L, and suppose that F exhibits a rational preference ranking relative to X and t. X's preferences might exhibit certain other features also. Suppose, for example, that X Eq b, a, c, t, α, where a, b, and c are all in F, and that he is indifferent as between a and d, where d is in F. Then very likely X Eq b, d, c, t, α also. Similarly if X Eq b, a, c, t, α and X Indiff b, d, t then X Eq d, a, c, t, α. Again, we should not *require* that these conditions hold of X's preferences. But it is not unreasonable to think that these and similar conditions would obtain for many F, X, and t.

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[3] See, e.g., H. Weyl, *Das Kontinuum* (Leipzig, 1918); S. C. Kleene, *Introduction to Metamathematics* (New York: Van Nostrand, 1952); and P. Lorenzen, *Einführung in die Operative Logik und Mathematik* (Berlin-Göttingen-Heidelberg: Springer, 1955).

Let us define directly the notion that F exhibits a *rational preference pattern* relative to X and t.[4]

'F RPP X, t' abbreviates '$(F$ RPR $X, t \,.\, (\alpha)(\beta)(a)(b)(c)(d)((Fa \,.\, Fb \,.\, Fc \,.\, Fd \,.\, {\sim}a = b \,.\, {\sim}a = c \,.\, {\sim}a = d \,.\, {\sim}b = c \,.\, {\sim}b = d \,.\, {\sim}c = d) \supset (((X \text{ Indiff } a, b, t \,.\, X \text{ Indiff } a, c, t) \supset X \text{ Eq } a, b, c, t, \alpha) \,.\, ((X \text{ Eq } b, a, c, t, \alpha \,.\, X \text{ Indiff } a, d, t) \supset X \text{ Eq } b, d, c, t, \alpha) \,.\, ((X \text{ Eq } b, a, c, t, \alpha \,.\, X \text{ Indiff } b, d, t) \supset X \text{ Eq } d, a, c, t, \alpha) \,.\, ((X \text{ Eq } b, a, c, t, \alpha \,.\, X \text{ Indiff } c, d, t) \supset X \text{ Eq } b, a, d, t, \alpha) \,.\, (X \text{ Prfr } a, c, t \supset (X \text{ Indiff } b, c, t \equiv X \text{ Eq } b, a, c, t, 0)) \,.\, (X \text{ Prfr } a, c, t \supset (X \text{ Indiff } a, b, t \equiv X \text{ Eq } b, a, c, t, 1)) \,.\, ((X \text{ Prfr } a, c, t \,.\, {\sim}X \text{ Prfr } b, a, t \,.\, {\sim}X \text{ Prfr } c, b, t) \supset (\text{E}\gamma)(X \text{ Eq } b, a, c, t, \gamma \,.\, (\delta)(X \text{ Eq } b, a, c, t, \delta \supset \delta = \gamma))).\ ((X \text{ Prfr } a, b, t \,.\, X \text{ Prfr } b, c, t \,.\, X \text{ Prfr } c, d, t \,.\, {\sim}\alpha = 0 \,.\, {\sim}\beta = 1) \supset (((X \text{ Eq } b, a, d, t, \alpha \,.\, X \text{ Eq } c, a, d, t, \beta) \equiv (X \text{ Eq } c, b, d, t, (\beta/\alpha) \,.\, X \text{ Eq } b, a, c, t, ((\alpha - \beta)/(1 - \beta)))) \,.\, ((X \text{ Eq } b, a, d, t, \alpha \,.\, X \text{ Eq } c, b, d, t, (\beta/\alpha)) \equiv (X \text{ Eq } c, a, d, t, \beta \,.\, X \text{ Eq } b, a, c, t, ((\alpha - \beta)/(1 - \beta)))).\ ((X \text{ Eq } b, a, d, t, \alpha \,.\, X \text{ Eq } b, a, c, t, ((\alpha - \beta)/(1 - \beta))) \equiv (X \text{ Eq } c, a, d, t, \beta \,.\, X \text{ Eq } c, b, d, t, (\beta/\alpha))))))))$'.

First we require that F be an RPR for X at t. Next we require that if X is indifferent as between suitable a and b at t, and also as between a and c at t, then he equates a with the combination of b and c for *any* α at t. The next three conditions are obvious conditions concerning indifference, in effect that names of sentences as between which X is indifferent at t are substitutable for each other in atomic contexts containing 'Eq'. The next two conditions are concerned with 0 and 1. Suppose X prefers a to c at t but is indifferent as between b and c. Then $(\text{u}(b) - \text{u}(c)) = 0$, $(\text{u}(a) - \text{u}(c)) > 0$, and hence the ratio of these two is 0. But if X is indifferent as between b and c, X then presumably would equate b with the combination of a and c to degree 0 at t. And conversely, if this latter, X preferring a to c at t, X then presumably would be indifferent as between b and c at t. The condition concerning 1 is similar.

If X prefers a to c at t then $(\text{u}(a) - \text{u}(c)) \neq 0$. Hence for any b, if X does not prefer b to a at t nor c to b at t, we may require that X equate b with the combination of a and c at t to one and only one degree γ.

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[4] Cf. Davidson, McKinsey, and Suppes, *op. cit.*, p. 156.

Finally, consider the situation where X prefers a to b at t, b to c at t, and c to d at t. Think of α, β, etc., as ratios of preference as described above. Let $\alpha = ((\mathrm{u}(b) - \mathrm{u}(d))/(\mathrm{u}(a) - \mathrm{u}(d)))$ and $\beta = ((\mathrm{u}(c) - \mathrm{u}(d))/(\mathrm{u}(a) - \mathrm{u}(d)))$. Then $((\mathrm{u}(c) - \mathrm{u}(d))/(\mathrm{u}(b) - \mathrm{u}(d))) = (\beta/\alpha)$ and $((\mathrm{u}(b) - \mathrm{u}(c))/(\mathrm{u}(a) - \mathrm{u}(c))) = ((\alpha - \beta)/(1 - \beta))$, and conversely. Thus the first of the last three conditions may reasonably hold. The other two conditions may hold similarly.

Although these conditions are eminently reasonable, the experimenter E in no way imposes them upon X. X's preferences and equatings may be erratic in the extreme or inconsistent in various ways, as we have already noted. E is concerned merely to observe and record his findings, and in this way to establish whether F RPP X, t or not. Of course X may wish to note his own inconsistencies, he may be grateful when they are pointed out to him, and so on. In this way E may be helpful to him, enabling him to establish a rational order among the sentences of F.

We have the following theorems.

TB1. $\vdash (F \text{ RPP } X, t \,.\, G \text{ RPP } X, t \,.\, \sim(F \cap G) = \Lambda) \supset (F \cap G) \text{ RPP } X, t.$

TB2. $\vdash (F \cup G) \text{ RPP } X, t \,.\, \sim F = \Lambda) \supset F \text{ RPP } X, t.$

C. THE ADEQUACY THEOREM

We now consider the problem of setting up a suitable metric for the measurement of degrees of acceptance. The fundamental theorem here is an adaptation of that of von Neumann and Morgenstern. Following them in essentials, we note that real numbers may be assigned to sentences of some suitable virtual class F so as to preserve and exhibit the structure of a rational preference pattern. This assignment or metric will have the usual characteristic of *interval* scales of measurement that it is unique once an interval and point of origin have been chosen.

An *interval* scale of measure, it will be recalled, is one with no fixed or "natural" zero or unit interval. The measurement of time, for example, is usually given by an interval measure. A certain time is picked out and other times are measured by their "distance," so to speak, from the given time. There are, of course, other types of scales

of measurement in common use in the sciences, for example, *ratio* scales (for mass or length), *absolute* scales (for the cardinality of classes), and *ordinal* scales (Beaufort wind scale). By means of an ordinal scale we locate relative position only. In a ratio scale we have a fixed zero (zero mass or zero length) but an arbitrary unit or interval, whereas an absolute scale has a fixed zero and a fixed unit.

Given any virtual class F, we may define within the metalanguage here a three-placed functional constant 'ϕ_F' taking sentences of F, persons X, and times t as arguments and having real numbers between and including 0 and 1 as values such that

TC1. $\vdash F\,\mathrm{RPP}\,X, t \supset (a)(b)(c)(\alpha)((Fa\,.\,Fb\,.\,Fc\,.\sim a = b\,.\sim a = c\,.\sim b = c\,.\,0 \leqq \alpha \leqq 1) \supset ((X\,\mathrm{Prfr}\,a, b, t \equiv \phi_F(a, X, t) > \phi_F(b, X, t))\,.\,(X\,\mathrm{Eq}\,a, b, c, t, \alpha \equiv \phi_F(a, X, t) = ((\alpha\,.\,\phi_F(b, X, t)) + ((1 - \alpha)\,.\,\phi_F(c, X, t))))))$.

(The '.' as between expressions for real numbers is to express multiplication. Its ambiguous use in this way and as a truth-functional connective will always be clear from the context.)

Further, given any two such functional constants, say 'ϕ_F' and 'ψ_F', i.e., any two constants for which *TC1* holds, we have that

TC2. $\vdash F\,\mathrm{RPP}\,X, t \supset (\mathrm{E}\beta)(\mathrm{E}\gamma)(\beta > 0\,.\,(a)(Fa \supset \phi_F(a, X, t) = ((\beta\,.\,\psi_F(a, X, t)) + \gamma)))$.

These two theorems together give the effect of the metric needed. *TC1* establishes that ϕ_F has the desired property, and *TC2* that this function is unique to within the specification of an interval and point of origin. In view of *TC1* we can identify $\phi_F(a, X, t)$ with the "utility" of a for X at t as discussed informally above.

The proofs of these two theorems involve some mathematical technicalities which need not concern us here.[5]

Hereafter we may let 'ϕ_F' and 'ψ_F' stand for any function of the kind described satisfying *TC1* and *TC2*.

D. DEGREE OF ACCEPTANCE

In terms of 'ϕ_F' let us now consider how we may best introduce the notion of degree of acceptance.

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[5] See von Neumann and Morgenstern, *op. cit.*, pp. 618–28.

Suppose that the sentence a is in F and that F RPP X, t. We might then say that X accepts a of F at t to the degree α if and only if $\phi_F(a, X, t) = \alpha$. This would be to identify outright the degree of acceptance of a sentence with its ϕ_F value. This seems not unreasonable. Hence we might let

'X Acpt a, F, t, α' abbreviate '$(F\, a \,.\, F$ RPP $X, t \,.\, \phi_F(a, X, t) = \alpha)$'.

The definiendum reads 'X accepts the sentence a of F at t to degree α'. For the purposes of this definition we must have of course some ϕ_F already available, but we know that such a function is forthcoming by *TC1* above.

It might be desirable for some purposes to require the scales for degree of acceptance to range from 0 to 1. According to the definition just given, different virtual classes of sentences would ordinarily have different end-points. One such class might have a scale ranging, say, from $\frac{2}{5}$ to $\frac{3}{5}$, whereas another might have a scale ranging from -17 to 922. In order to avoid this and to bring uniformity to the scales for different virtual classes of sentences, it will be convenient to require the scales to range between 0 and 1 as well as always to take on 0 and 1 as end-values.

For this we need the notion of being the *maximum* value of ϕ_F for X at t and of being the *minimum* value of ϕ_F for X at t. We let

'α Max ϕ_F, X, t' abbreviate '$(F$ RPP $X, t \,.\, (\mathrm{E}a)(F\, a \,.\, \phi_F(a, X, t) = \alpha)$ $.\, (b)(F\, b \supset \phi_F(b, X, t) \leq \alpha))$',

and

'α Min ϕ_F, X, t' abbreviate '$(F$ RPP $X, t \,.\, (\mathrm{E}a)(F\, a \,.\, \phi_F(a, X, t) = \alpha)$ $.\, (b)(F\, b \supset \phi_F(b, X, t) \geq \alpha))$'.

We may now define the notion that X accepts a of F at t to the degree α, where α takes on the end values 0 and 1, as follows.

'X Acpt a, F, t, α' abbreviates '$(F\, a \,.\, (\mathrm{E}\beta)(\mathrm{E}\gamma)(\beta$ Max $\phi_F, X, t \,.\, \gamma$ Min $\phi_F, X, t \,.\, \alpha = ((\phi_F(a, X, t) - \gamma)/(\beta - \gamma))))$'.

For this second notion of degree of acceptance, the values for given sentences as arguments (relative, of course, to X and t) are constructed by picking out the maximum and minimum values and subtracting

them in order to form the denominator. The numerator then consists of the actual ϕ_F value minus the minimum. In this way all degrees of acceptance are telescoped, so to speak, into the interval 0 to 1.

Suppose 29 is the maximum value of ϕ_F relative to X and t and 19 the minimum. And suppose $\phi_F(a, X, t) = 27$ for a specific sentence a of F. The degree of acceptance of a relative to X and t is then to be $\frac{4}{5}$ according to this definition. If $\frac{2}{3}$ is the maximum and $\frac{1}{3}$ the minimum value of ϕ_F relative to X and t and $\phi_F(a, X, t) = \frac{7}{16}$, then the degree of acceptance of a relative to X and t is $\frac{5}{16}$.

The following theorems are immediate concerning this second notion.

TD1. $\vdash F \text{ RPP } X, t \supset (\text{E}\alpha)(\alpha \text{ Max } \phi_F, X, t \,.\, (\beta)(\beta \text{ Max } \phi_F, X, t \supset \alpha = \beta))$.

TD2. $\vdash F \text{ RPP } X, t \supset (\text{E}\alpha)(\alpha \text{ Min } \phi_F, X, t \,.\, (\beta)(\beta \text{ Min } \phi_F, X, t \supset \alpha = \beta))$.

TD3. $\vdash (F a \,.\, F \text{ RPP } X, t) \supset (\text{E}\alpha)(X \text{ Acpt } a, F, t, \alpha \,.\, (\beta)(X \text{ Acpt } a, F, t, \beta \supset \beta = \alpha))$.

TD4. $\vdash X \text{ Acpt } a, F, t, \alpha \supset 0 \leqq \alpha \leqq 1$.

TD1 and *TD2* embody existence and uniqueness laws for the maximum and minimum of any virtual class which exhibits a rational preference pattern. *TD3* is an existence and uniqueness law for the degree of acceptance, and *TD4* states that the degrees of acceptance must always be between 0 and 1. Also

TD5. $\vdash X \text{ Acpt } a, F, t, \alpha \supset ((\text{E}b) X \text{ Acpt } b, F, t, 1 \,.\, (\text{E}b) X \text{ Acpt } b, F, t, 0)$.

This theorem brings out the fact that if X Acpt a, F, t, α in the second sense then there is a sentence in F which X accepts to maximal degree 1 and one which he accepts to minimal degree 0.

We have then these two notions of degree of acceptance. Both notions should be useful, but in the sequel we shall employ the second exclusively. It is perhaps better always to take the maximal value as 1. This would make it especially easy to locate the relative position of a sentence in F given its degree of acceptance. Also comparison with the paradigmatic use of confirmation is then immediate.

We have in view of *TC1* the following.

TD6. $\vdash (X \text{ Acpt } a, F, t, \alpha \,.\, X \text{ Acpt } b, F, t, \beta) \supset (X \text{ Prfr } a, b, t \equiv \alpha > \beta)$.

TD7. $\vdash (X \text{ Acpt } a, F, t, \alpha \,.\, X \text{ Acpt } b, F, t, \beta) \supset (X \text{ Indiff } a, b, t \equiv \alpha = \beta)$.

TD8. $\vdash (X \text{ Acpt } a, F, t, \alpha \,.\, X \text{ Acpt } b, F, t, \beta \,.\, X \text{ Acpt } c, F, t, \gamma \,.\, {\sim} a = b \,.\, {\sim} b = c \,.\, {\sim} a = c \,.\, 0 \leqq \delta \leqq 1) \supset (X \text{ Eq } a, b, c, t, \delta \equiv \alpha = ((\delta \,.\, \beta) + ((1 - \delta) \,.\, \gamma)))$.

E. SOME ACCEPTANCE PATTERNS

Now that the notion 'Acpt' is available, several different types of acceptance patterns may be distinguished. These will have some affinity with the normal preference rankings of (II, E).

Suppose F exhibits a RPP relative to X at t and contains at least one sentence a together with (*tilde* a). If for every such a and (*tilde* a) and for every real number α, X accepts a of F at t to the degree α if and only if he accepts (*tilde* a) to the degree $(1 - \alpha)$, then F may be said to exhibit a *normal acceptance pattern* (NAP) *relative to tilde*, X, *and* t. Thus we may let

'F NAP *tilde*, X, t' abbreviate '$((\mathrm{E}a)(F a \,.\, F(\textit{tilde } a)) \,.\, F \text{ RPP } X, t \,.\, (a)(\alpha)((F a \,.\, F(\textit{tilde } a)) \supset (X \text{ Acpt } a, F, t, \alpha \equiv X \text{ Acpt } (\textit{tilde } a), F, t, (1 - \alpha))))$'.

If F exhibits a normal acceptance pattern in this sense, X then accepts to the degree α sentences of F beginning with *tilde* in the normal way, i.e., in accord with the normal or logically correct use of *tilde*. Recall (§H of Chapter I) that α is the confirmation of a if and only if $(1 - \alpha)$ is the confirmation of (*tilde* a) on suitable e. This circumstance guides us paradigmatically in framing this definition.

Note that

TE1. $\vdash (F \text{ NAP } \textit{tilde}, X, t \,.\, F a \,.\, F(\textit{tilde } a) \,.\, {\sim} X \text{ Indiff } a, (\textit{tilde } a), t \,.\, X \text{ Acpt } a, F, t, \alpha) \supset {\sim} X \text{ Acpt } (\textit{tilde } a), F, t, \alpha$.

We may also introduce the notion of a normal acceptance pattern for *vee*. Let F exhibit a RPP relative to X at t and be such as to contain

some a and some b together with (a *vee* b) and (a *dot* b). Let α be X's degree of acceptance of a at t, β of b at t, γ of (a *vee* b) at t, and δ of (a *dot* b) at t. For all such a and b we require that $\gamma = (\alpha + \beta - \delta)$. Comparison with the law *TH11* of Chapter I justifies the choice of this sum for γ. Thus we may let

'F NAP *vee*, X, t' abbreviate '((Ea)(Eb)($F\,a$. $F\,b$. F (a *vee* b) . F (a *dot* b)) . F RPP X, t . (a)(b)(α)(β)(γ)(δ)((X Acpt a, F, t, α . X Acpt b, F, t, β . X Acpt (a *dot* b), F, t, δ . X Acpt (a *vee* b), F, t, γ) $\supset \gamma = (\alpha + \beta - \delta)$))'.

Here again if F exhibits a normal acceptance pattern for *vee* relative to X and t, the degrees of X's acceptances of sentences of F at t accord with the logically correct degrees of confirmation involved.

The notion of a normal acceptance pattern for conjunction may be introduced in a very similar way. In fact we may let

'F NAP *dot*, X, t' abbreviate --- same definiens as in the definition of 'NAP' for *vee* but with '$\delta = (\alpha + \beta - \gamma)$' replacing '$\gamma = (\alpha + \beta - \delta)$' ---.

But we should then note immediately that the two definientia of the definitions of 'NAP' for *vee* and *dot* are equivalent, so that

TE2. $\vdash F$ NAP *vee*, X, $t \equiv F$ NAP *dot*, X, t.

Hence an NAP for either of these constants is simultaneously an NAP for the other.

For logical theorems, theorems, truths, and confirmation, we may introduce normal acceptance patterns as follows.

'F NAP *LogThm*, X, t' abbreviates '((Ea)(Eb)($F\,a$. $F\,b$. LogThm a . $\sim$Tr b) . (a)(b)(α)(β)((LogThm a . $\sim$Tr b . X Acpt a, F, t, α . X Acpt b, F, t, β) $\supset \alpha > \beta$))'.

If X's degree of acceptance of logical theorems of F is consistently greater than his degree of acceptance of falsehoods of F, clearly some kind of normalcy is being exhibited. And similarly for Thm and Tr.

Normal acceptance patterns for confirmation are as follows, where

we assume for the moment (as in (II, E)) that the metalanguage contains confirmation theory.

'F NAP *Conf*, X, t, e' abbreviates '(F RPP X, t . Sent e . $\sim$LFIs e . $(a)(\alpha)(F\,a \supset (\mathrm{c}(a, e) = \alpha \equiv X$ Acpt $a, F, t, \alpha)))$'.

In the important case, of course, e will be the total relevant evidence available to X at the time.

Let us now consider some rather tentative normal acceptance patterns for the rules of inference *MP* and *Gen* (*Modus Ponens* and the Rule of Generalization). What would it mean to say that a virtual class of sentences F exhibits a normal acceptance pattern, say, for *Modus Ponens*? Let us assume, of course, that F contains the necessary sentences, i.e., at least some a and b with (a *hrsh* b). Suppose X accepts a of F at t to degree α and (a *hrsh* b) of F at t to degree $\beta \geqq \alpha$. Then he should presumably accept b of F at t to at least degree α also. Similarly, if X accepts (a *hrsh* b) to degree α and a to a degree $\beta \geqq \alpha$, then he should presumably accept b to at least degree α also. If these circumstances hold for all a, b, and (a *hrsh* b) in F, it seems reasonable to say that F exhibits a normal acceptance pattern for *MP* relative to X at t. (Note that X's acceptance of b has the *lower* degree of the degrees of his acceptances of a and (a *hrsh* b) if they are different.) Thus we may perhaps let

'F NAP *MP*, X, t' abbreviate '((Ea)(Eb)($F\,a$. $F\,b$. F (a *hrsh* b)) . $(a)(b)(\alpha)$(($F\,a$. $F\,b$. F (a *hrsh* b) . ((X Acpt a, F, t, α . (Eβ)($\beta \geqq \alpha$. X Acpt (a *hrsh* b), F, t, β)) v (X Acpt (a *hrsh* b), F, t, α . (Eβ)($\beta \geqq \alpha$. X Acpt a, F, t, β)))) $\supset$ (Eβ)($\beta \geqq \alpha$. X Acpt b, F, t, β)))'.

Similarly

'F NAP *Gen*, X, t' may abbreviate '((Ea)(Eb)($F\,a$. $F\,b$. b Gen a) . $(a)(b)(\alpha)$(($F\,b$. b Gen a . X Acpt a, F, t, α) $\supset$ (Eβ)($\beta \geqq \alpha$. X Acpt b, F, t, β)))',

where 'b Gen a' expresses that b is a result of universally quantifying a. Hence also

'F NAP *IC*, X, t' may abbreviate '(F NAP *MP*, X, t . F NAP *Gen*, X, t)'.

The definienda of these last two definitions are to be read respectively 'F exhibits a normal acceptance pattern for *Gen*, the Rule of Generalization, relative to X at t', and 'X exhibits a normal acceptance pattern for IC, the relation of *immediate consequence*, relative to X at t'.

Perhaps a stricter meaning for 'NAP' relative to *MP* should be introduced, in roughly the following way. Let F contain a, b, and (a *hrsh* b) as above. Suppose (i) X Acpt a, F, t, α and (ii) X Acpt (a *hrsh* b), F, t, β. Let f be a real function of two variables such that where $\mathrm{c}(a, e) = \alpha$ (for suitable e) and $\mathrm{c}((a\ hrsh\ b), e) = \beta$, $\mathrm{c}(b, e) = \mathrm{f}(\alpha, \beta)$. Presumably such a function can be defined. Then we could stipulate, where (i) and (ii), that X Acpt $b, F, t, \mathrm{f}(\alpha, \beta)$. A definition incorporating this requirement would in effect involve the paradigmatic use of confirmation. Such a definition would provide a stricter definition of 'NAP' for *MP* and would perhaps be more useful than the definition given.

The definitions of these various kinds of NAP's are put forward as mere suggestions. Some of them may perhaps be improved. No doubt there are many interesting alternatives also, as well as further kinds, to be discriminated. All of them, like the definitions of the rational rankings and patterns above, attempt to set up certain desirable or ideal patterns which may or may not be realized *in concreto*. But even if not, they may be useful as ideals or paradigms with which patterns actually realized may be compared or contrasted.

F. THE SOCIAL GROUP

Many of the definitions of the preceding section, as well as many definitions to be given subsequently, we shall wish to have in a more general form. The definitions of the preceding section are all relative to a person X. It will be convenient to generalize these definitions to hold for *virtual classes of persons* as well.

As described essentially in (I, B) we may introduce abstracts of the form '$Y \ni B$' where Y is a variable for the users of L. Where X is a variable or constant for a user, we have then

$\vdash Y \ni B\ X = A$, where (etc., essentially as in the Rule of Abstraction of (I, B)).

Just as above, the notion of being a free occurrence of a variable X in A must now be phrased so as to assure that no occurrences of X in a context '$X \ni A$' are regarded as free occurrences.

We let 'I', 'J', and 'K', with or without accents or numerical subscripts, abbreviate any abstracts containing no free variables of the kind just introduced.[6]

We can develop now a Boolean algebra of virtual classes of persons analogous to that suggested in (I, C). To simplify, let us use '$\cup$', '$\cap$', etc., ambiguously hereafter to stand for either the Boolean sum, etc., of virtual classes of expressions or of persons. It will always be clear from the context which is intended.

The definitions of the preceding section may now be extended to virtual classes of persons. For example, we may say that the sentences of F exhibit a normal preference ranking for *tilde* at t relative to a (nonnull) virtual class of persons I as follows.

'F NAP *tilde*, I, t' abbreviates '$(\sim I = \Lambda \,.\, (X)(I\, X \supset F$ NAP *tilde*, $X, t))$'.

Clearly we have the following theorems.

TF1. $\vdash (F$ NAP *tilde*, $I, t \,.\, F$ NAP *tilde*, $J, t) \supset F$ NAP *tilde*, $(I \cup J), t$.

TF2. $\vdash (F$ NAP *tilde*, $I, t \,.\, F$ NAP *tilde*, $J, t \,.\, \sim(I \cap J) = \Lambda) \supset F$ NAP *tilde*, $(I \cap J), t$.

TF3. $\vdash (F$ NAP *tilde*, $(I \cup J), t \,.\, \sim I = \Lambda \,.\, \sim J = \Lambda) \supset (F$ NAP *tilde*, $I, t \,.\, F$ NAP *tilde*, $J, t)$.

Similar generalized definitions may be given for the other patterns and rankings of the preceding sections and of Chapter II also. Such definitions provide a means of extending the notions concerned to social groups. The groups may be suitable samples of the whole population but only in rare cases will they themselves constitute the whole

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[6] Note that 'F', 'G', etc., are used in effect for virtual classes of expressions of L, whereas 'I', 'J', etc., here are used to stand for virtual classes of *persons*. No notation for virtual classes of *times* will be needed, the time-variables themselves ranging over time-stretches of any duration desired.

population. There are difficulties involved in generalizing the foregoing definitions to the whole population, which cannot be considered here.[7]

G. SOME ALTERNATIVE BASES

The notion of degree of acceptance has been introduced above in terms of a primitive 'Eq', utilizing a utility theory somewhat in the manner of von Neumann and Morgenstern. The experimental meaning of 'Eq' has been indicated only roughly. Exact experimental meanings may be given in specific contexts. Much work by statisticians, psychologists, and social scientists generally is being done on what is essentially the problem of giving such experimental meanings. Also for some purposes it might be desirable to modify some of the formal requirements in the definition of 'RPP'. But these are matters that cannot be considered further here. For the sequel we need presuppose only a notion of degree of acceptance howsoever defined. The definition suggested is merely one out of many possible. But whether it is the best definition or even a wholly satisfactory one must await further work in or upon decision theory and its experimental basis. We must be content here merely to use decision theory for philosophical purposes, not to alter or improve it.

It was suggested in (II, F) that 'Prfr' should perhaps be reconstrued as 'Prfr_e', i.e., as a relation of "rational" preference governed by given evidence e. Similarly 'Eq' should perhaps be reconstrued to take explicit account of the evidence e. Of course this may easily be done by a suitable change in the preceding material. It is not clear that this is essential, however, because 'Eq' may always be interpreted, if desired, as involving implicit reference to the total evidence available at the time, available to X or E or both.

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[7] Cf., for example, K. J. Arrow, *Social Choice and Individual Values* (New York and London: J. H. Wiley and Co., 1951), for analogous difficulties concerning other types of definition.

IV

SUBJECTIVE QUASI-INTENSIONS AND COINTENSIVENESS

In terms of the notion of degree of acceptance a quantitative theory of notions closely akin to that of *subjective intension* may be developed. The subjective intension or connotation of a term or name has traditionally been thought to consist of "those properties which in the mind of any individual are associated with the name in such a way that they are normally called up in idea when the name is used."[1] We need not ponder as to the precise significance of this traditional notion, which is no doubt obscure. Instead, we shall develop in this chapter a theory of notions sufficiently close to this traditional one to justify the terminology.

To avoid making any identification, however, between the notions defined here and the traditional one or ones, we shall speak for the present of subjective *quasi*-intensions, of *quasi*-cointensiveness, and so on. This terminology, further, will help to emphasize that we are not dealing with intensions in the sense of attributes, class-concepts, and the like, but only with virtual classes of expressions as in (I, E–G).[2]

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[1] John Neville Keynes, *Formal Logic*, 2nd. ed. (London: Macmillan, 1887), p. 24.

[2] On attributes, class-concepts, and the like, see, e.g., B. Russell, *The Principles of Mathematics*, 2nd. ed. (New York: Norton, 1938); G. Frege, *Philosophical Writings*,

A systematic theory of subjective quasi-intensions presumably could not be developed without the kind of groundwork laid in the previous chapters. Such a theory surely must presuppose syntax and semantics. Further, it is pragmatical, involving fundamental reference to the users of the language. Also there is fundamental reference to time, subjective quasi-intensions varying with time. Finally, in order to develop a quantitative theory, some suitable adaptation of modern utility and decision theory seems needed, such as that of the preceding chapter.

Tentative steps toward developing a theory of subjective quasi-intensions were taken in *Toward a Systematic Pragmatics*. But the theory there was based merely upon a nonquantitative notion of acceptance. In this present chapter a much more refined theory based upon the quantitative notion of degree of acceptance is developed.

In §A several kinds of subjective quasi-intensions of degree α are introduced and several kinds of *intersubjective* and *intertemporal* quasi-intensions are defined in §B. The notion of a quasi-intension for a *social group* is discussed in §C and various relations of *quasi-cointensiveness* are introduced in §D. In §E we return to the notion of a *quasi-proposition* and consider a concept of *degree of rational belief*. In §F an epistemological relation of being *more acceptable than* is defined and in §G an outline of a logic of some *further epistemological notions* is given. The possibility of a quantitative pragmatics *based only on syntax* is discussed in §H. Finally, in §I, some reflections are put forward as to how the preceding material concerning the systems *L* may perhaps be extended to *natural language*.

A. SUBJECTIVE QUASI-INTENTIONS OF DEGREE α

In this section several closely related notions of subjective quasi-intension are defined, and some examples are given. There are

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ed. by P. Geach and M. Black (Oxford: Blackwell, 1952); A. Church, *Introduction to Mathematical Logic I*, esp. Chapter 1; and R. Carnap, *Meaning and Necessity*. Cf. also the author's "On the Frege-Church Theory of Meaning," forthcoming in *Philosophy and Phenomenological Research*. On the more traditional views, see in particular A. Arnaud and P. Nicole, *La Logique, ou l'Art de Penser* (Paris: 1662) and J. S. Mill, *A System of Logic* (London: 1843). Cf. also I. M. Bocheński, *A History of Formal Logic* (Notre Dame: University of Notre Dame Press, 1961), and W. Kneale and M. Kneale, *The Development of Logic* (Oxford: Clarendon Press, 1962).

subjective *analytic* quasi-intensions of degree α for individual constants, subjective *veridical* quasi-intensions, subjective *synthetic* and *theoremic* quasi-intensions, and also another kind which we shall call subjective *contingent* quasi-intensions, all of degree α, for individual constants. For each of these types there will be two sorts, depending upon the way in which the degree α is determined. There will be a similar subdivision for one-place predicate constants, subjective analytic (comprehensional) quasi-intensions of degree α, subjective veridical (comprehensional) quasi-intensions, and so on. And similarly for two-place predicate constants, and so on. Here likewise there will be two sorts for each, depending upon how the degree α is determined.

Subjective analytic quasi-intensions of degree α for individual constants are to be relative to the person X, as has already been suggested. The use of the word 'subjective' in fact is intended to emphasize this relativeness, and to indicate that we are no longer dealing with objective quasi-intensions in the sense of Chapter I. And also subjective quasi-intensions should clearly be relative to time, as we have already noted. The subjective quasi-intension of a for X at t_1 may well differ from that at t_2, for t_1 wholly distinct from t_2. Subjective quasi-intensions should also be relative to a virtual class F of sentences in which the experimenter E is interested at the time. The relativization to an F in this way provides an important factor of experimental control, and will appear in almost all of the definitions to be given.

Let a for the moment be a primitive or defined individual constant of L, and let F have as members a good many sentences containing a. Let b be such a sentence and suppose it is accepted by X at time t to the degree α. Further, suppose that b is an analytic sentence of L. Each such sentence b may be regarded as ascribing a certain property to the individual designated by a, i.e., as saying that this individual is a member of a certain virtual class. (For example, if b is of the form of a law of excluded middle, then the corresponding virtual class is the universal virtual class of individuals.) Let us consider now the virtual class of *all* abstracts standing for such virtual classes. Clearly this is an important virtual class. (It is closely related in fact to the virtual classes considered in (I, G).) Each abstract in it is determined by a sentence b of F which is analytic and is accepted by X at t to the degree α. Such a virtual class of abstracts is subjective, being relativized to

the person X. And because each sentence b is analytic, the members of such a virtual class are wholly determined by the underlying *logic* of L, so to speak, analytic sentences being just logical truths. In these various respects, then, such virtual classes of abstracts are akin to what have traditionally been called 'subjective intensions'. The proposal here is not to identify the two, but rather to regard such virtual classes of abstracts as subjective quasi-intensions.

We shall distinguish two senses in which certain virtual classes of abstracts may be regarded as subjective analytic quasi-intensions of individual constants, a narrower and a wider sense. The virtual classes of abstracts discussed in the preceding paragraph are to be subjective analytic quasi-intensions in the narrower sense.

The notion of being a subjective analytic quasi-intension of *exactly the degree* α of an individual constant a relative to X, F, and t may be defined as follows.

'G $\mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{InCon}}$ a, X, F, t, α' abbreviates '(InCon a . $G = c \ni (\mathrm{E}b)(\mathrm{E}d)(\mathrm{E}e)(c = (d \cap \mathit{invep} \cap e)$. $b \, \mathrm{S}^a_d \, e$. Anlytc b . X Acpt b, F, t, α))'.

(The superscript '1' in the definiendum is to suggest that a notion of subjective analytic quasi-intension *in the first sense* is being defined, another to follow.) Roughly speaking, according to this definition, a virtual class of expressions G is a $\mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{InCon}}$ of exactly the degree α of an individual constant a relative to X, F, and t, if G is the virtual class of all abstracts c where X accepts to the degree α an analytic sentence of F stating in effect that the object designated by a is a member of the virtual class for which c stands.

Let us consider an example. Suppose a is the primitive InCon 'a', and suppose X accepts at t the analytic sentence '$\mathrm{a} = \mathrm{a}$' of F to the degree α. Consider now the abstract '$x \ni x = \mathrm{a}$'. Clearly this is of the form $(d \cap \mathit{invep} \cap e)$, where b, i.e., '$\mathrm{a} = \mathrm{a}$', bears S^a_d to e, i.e., to '$x = \mathrm{a}$' (a being 'a' and d being 'x'), and hence is a member of X's subjective analytic quasi-intension of degree exactly α of 'a' relative to F and t. Suppose X also accepts to the degree α the analytic sentence '$\sim \mathrm{a} = \mathrm{b}$' at t, this sentence also being a member of F. Then the abstract '$x \ni \sim x = \mathrm{b}$' is also a member of X's subjective analytic quasi-intension of degree exactly α of 'a' relative to F and t. In general, let '--a--' be an analytic

sentence of F accepted by X at t to the degree α. Then the abstract '$x\ni$--x--' is a member of the $\mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{InCon}}$ of X of 'a' relative to F and t.

Note that in place of '$b\ \mathrm{S}^a_d\ e$' in the definiens of this definition we could have written 'SentFuncOne e,d . $b = (c \frown a)$', where 'SentFuncOne e,d' expresses that e is a sentential function of the one variable d, i.e., that e is a formula containing d as its only free variable (if any). This would entail ever so slight a change, however, in the meaning of the definiens, the sentences accepted by X being then of different form.

The following existence and uniqueness laws are immediate.

TA1. $\vdash ((\mathrm{E}b)(\mathrm{E}c)(\mathrm{E}d)(b\ \mathrm{S}^a_c\ d$. Anlytc b . X Acpt $b, F, t, \alpha)$. G Subj-$\mathrm{AnlytcQuasiInt}^1_{\mathrm{InCon}}\ a, X, F, t, \alpha) \supset \sim G = \Lambda$.

TA2. $\vdash (G\ \mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{InCon}}\ a, X, F, t, \alpha$. H SubjAnlytc-$\mathrm{QuasiInt}^1_{\mathrm{InCon}}\ a, X, F, t, \alpha) \supset G = H$.

Subjective analytic quasi-intensions of exactly degree α contain as members only abstracts determined, so to speak, by sentences accepted to just the degree α. Subjective analytic quasi-intensions of degree α in a *second sense* are to include all abstracts determined by sentences accepted to the degree α *or greater*. To allow such additional sentences to determine members of subjective analytic quasi-intensions seems a very natural extension. We add on, to speak loosely, to the subjective analytic quasi-intensions of degree α as defined above those abstracts determined by the appropriate sentences accepted to a degree greater than α. A subjective analytic quasi-intension in this second sense we may speak of as being of degree α *and upwards*, because the appropriate sentences determining members of it may be accepted by X to any degree from and including α up through degrees greater than α. Let us write '$\mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}$' for this notion. The formal definition is as follows.

'$G\ \mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}\ a, X, F, t, \alpha$' abbreviates '(InCon a . $(\mathrm{E}b)(\mathrm{E}c)(\mathrm{E}d)(b\ \mathrm{S}^a_c\ d$. Anlytc b . X Acpt $b, F, t, \alpha)$. $\sim(\mathrm{E}b)(\mathrm{E}c)(\mathrm{E}d)(\mathrm{E}\beta)(b\ \mathrm{S}^a_c\ d$. Anlytc b . $\beta < \alpha$. X Acpt $b, F, t, \beta)$. $G = c\ni(\mathrm{E}b)(\mathrm{E}d)(\mathrm{E}e)(\mathrm{E}\beta)(c = (d \frown \mathit{invep} \frown e)$. $b\ \mathrm{S}^a_d\ e$. Anlytc b . $\beta \geqq \alpha$. X Acpt $b, F, t, \beta))$'.

A virtual class of abstracts G is the subjective analytic quasi-intension of degree α and upwards of an individual constant a if it consists of all

appropriate abstracts determined by analytic sentences of F accepted by X to degree β, for $\beta \geqq \alpha$, provided F contains at least one analytic sentence containing a accepted by X to degree α and no such sentence accepted by X to a lower degree.

Concerning this notion we have the following existence and uniqueness laws.

TA3. $\vdash G\ \mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}\ a, X, F, t, \alpha \supset {\sim} G = \Lambda$.

TA4. $\vdash (G\ \mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}\ a, X, F, t, \alpha\ .\ H\ \mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}\ a, X, F, t, \alpha) \supset G = H$.

The following law gives the obvious condition under which a subjective analytic quasi-intension of degree exactly α is identical with one of degree α and upwards.

TA5. $\vdash (G\ \mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{InCon}}\ a, X, F, t, \alpha\ .\ H\ \mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}\ a, X, F, t, \alpha\ .\ {\sim}(\mathrm{E}b)(\mathrm{E}c)(\mathrm{E}d)(\mathrm{E}\beta)(b\ \mathrm{S}^a_c\ d\ .\ \mathrm{Anlytc}\ b\ .\ \beta > \alpha\ .\ X\ \mathrm{Acpt}\ b, F, t, \beta)) \supset G = H$.

In addition to the subjective analytic quasi-intensions of both kinds, we must also note the subjective quasi-intensions *based on L-equivalence* of both kinds. In particular

'$G\ \mathrm{SubjLEquivQuasiInt}^1_{\mathrm{InCon}}\ a, X, F, t, \alpha$' may abbreviate '$(\mathrm{InCon}\ a\ .\ G = b \ni (b\ \mathrm{LEquiv}_{\mathrm{InCon}}\ a\ .\ X\ \mathrm{Acpt}\ (b \frown id \frown a), F, t, \alpha))$'.

A virtual class of expressions G is the subjective quasi-intension based on L-equivalence of degree exactly α of an individual constant a for X relative to F and t, provided G is the virtual class of individual constants b such that b is L-equivalent with a and X accepts the sentence of their identity (this sentence being in F) to degree exactly α at t. The corresponding notion of degree α and upwards is definable in a similar way.

Some further notions are related to subjective analytic quasi-intensions essentially in the way in which the objective veridical, synthetic, and theoremic quasi-intensions of (I, G) are related to objective analytic quasi-intensions. And for each type, we will have two kinds depending upon how the numerical measure is determined. We may let

'$G\ \mathrm{SubjVerQuasiInt}^1_{\mathrm{InCon}}\ a, X, F, t, \alpha$',

etc., in obvious fashion, symbolize these various notions.

For each such notion, we have laws analogous to *TA1–TA5*, perhaps with suitable existence hypotheses.

Still a further notion is available here which has no analogue in the theory of Chapter I. We may let

'G SubjContQuasiInt$^1_{\mathrm{InCon}}$ a, X, F, t, α' abbreviate '(InCon a . G = $c\ni$(Eb)(Ed)(Ee)(c = ($d \frown inrep \frown e$) . b S^a_d e . X Acpt b, F, t, α))'.

We say that G is the subjective *contingent* quasi-intension for the individual constant a relative to X, F, and t to degree exactly α provided G is the virtual class of abstracts described. Here it is not required that b be an Anlytc, a Tr, or etc., but only that it be accepted by X to degree exactly α. In other words, b is contingent merely upon X's acceptance to the degree α at the time.

Subjective contingent quasi-intensions for individual constants of degree α and upwards may be defined similarly.

All of the preceding definitions of this section are concerned only with individual constants. For predicate constants we have suitable subjective and other quasi-intensions based on *comprehension*, as one would expect from (I, G). In fact we may let

'G SubjAnlytcQuasiInt$^1_{\mathrm{PredConOne}}$ a, X, F, t, α' abbreviate '(PredCon-One a . G = $b\ni$(Ec)(Ed)(PredConOne b . Vbl d . c = (d qu ($a \frown d$ $hrsh$ $b \frown d$)) . Anlytc c . X Acpt c, F, t, α))'.

Similarly we may define

'G SubjAnlytcQuasiInt$^2_{\mathrm{PredConOne}}$ a, X, F, t, α',

'G SubjVerQuasiInt$^1_{\mathrm{PredConOne}}$ a, X, F, t, α',

and so on, to express the appropriate notions. And similarly for PredConTwo's, and so on. Here also we have subjective *contingent* quasi-intensions for predicate constants of degree exactly α or α and upwards.

By way of an example, let us turn again to the primitive predicate constant 'P'. Suppose a is 'P' and the sentence '(x)(Px ⊃ (Px v Qx))', where 'Q' is another primitive one-place predicate constant, is in F and is accepted by X at t to the degree α. The abstract '$x\ni$(Px v Qx)' is then a member of G where

G SubjAnlytcQuasiInt$^1_{\mathrm{PredConOne}}$ a, X, F, t, α.

In general, if '(x)(Px ⊃ --x--)' is an analytic sentence in F accepted by

X at t to the degree α, then the abstract '$x \ni$--x--' is also in G. And similarly for the other kinds of intensions of PredConOne's.

The subjective analytic quasi-intensions for predicate constants are perhaps more interesting than those for individual constants, in the sense of having a more varied membership, just as the objective analytic quasi-intensions for predicate constants, as we have already observed, are rather more interesting than those for individual constants.

We note that the various subjective quasi-intensions of a constant are in general virtual subclasses of the corresponding objective quasi-intensions.

TA6. $\vdash ((G\ \mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{InCon}}\ a, X, F, t, \alpha \vee G\ \mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}\ a, X, F, t, \alpha) \,.\, H\ \mathrm{ObjAnlytcQuasiInt}_{\mathrm{InCon}}\ a) \supset G \subset H.$

TA7. $\vdash ((G\ \mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{PredConOne}}\ a, X, F, t, \alpha \vee G\ \mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{PredConOne}}\ a, X, F, t, \alpha) \,.\, H\ \mathrm{ObjAnlytcQuasiInt}_{\mathrm{PredConOne}}\ a) \supset G \subset H.$

And so on.

B. INTERSUBJECTIVE AND INTERTEMPORAL QUASI-INTENSIONS

In this section we consider several further types of quasi-intension. And within each type two subtypes are distinguished, depending upon how the numerical measure is determined.

An *intersubjective analytic quasi-intension* of a constant a, relative to the sentences of F and to a time t and of degree α or from α upwards, is also to be relative to a given social group of persons. The word 'intersubjective' is intended to suggest this. The intersubjective analytic quasi-intension of degree exactly α will contain as members just those abstracts common to all subjective analytic quasi-intensions of members of the given social group. Thus in effect the subjective analytic quasi-intension of degree exactly α is the logical product of the subjective analytic quasi-intensions of degree exactly α of the various members of that group, all relative of course to the same a, F, and t. We do not define the notion directly in this way but rather as follows.

'$G\ \mathrm{IntSubjAnlytcQuasiInt}^1_{\mathrm{InCon}}\ a, F, I, t, \alpha$' abbreviates '$(\mathrm{InCon}\ a \,.\, {\sim} I = \Lambda \,.\, G = c \ni (\mathrm{E}b)(\mathrm{E}d)(\mathrm{E}e)(c = (d \frown \mathit{inrep} \frown e) \,.\, b\ \mathrm{S}^a_d\ e \,.\, \mathrm{Anlytc}\ b \,.\, (X)(I\ X \supset X\ \mathrm{Acpt}\ b, F, t, \alpha)))$'.

Similarly

'G IntSubjAnlytcQuasiInt$^{2}_{\mathrm{InCon}}$ a, F, I, t, α' may abbreviate '(InCon a . $\sim I = \Lambda$. (Eb)(Ec)(Ed)(b S^{a}_{c} d . Anlytc b . (X)(I $X \supset X$ Acpt b, F, t, α)) . $\sim$(Eb)(Ec)(Ed)(Eβ)(b S^{a}_{c} d . Anlytc b . $\beta < \alpha$. (X)(I $X \supset X$ Acpt b, F, t, β)) . $G = c \ni$(Eb)(Ed)(Ee)(Eβ)($c = (d \cap \mathit{invep} \cap e)$. b S^{a}_{d} e . Anlytc b . $\beta \geqq \alpha$. (X)(I $X \supset X$ Acpt b, F, t, β)))'.

Intersubjective analytic quasi-intensions in the first sense, or of degree exactly α, consist just of abstracts determined by the appropriate analytic sentences accepted to degree exactly α by *all* members of the social group. Intersubjective analytic quasi-intensions in the second sense, or of degree α and upwards, consist of all abstracts determined by the appropriate analytic sentences accepted by all members of the social group to degree α and upwards.

The following theorems correspond with *TA1–TA5* but with the necessary additional hypotheses.

TB1. $\vdash$ ((Eb)(Ec)(Ed)(b S^{a}_{c} d . Anlytc b . (X)(I $X \supset X$ Acpt b, F, t, α)) . G IntSubjAnlytcQuasiInt$^{1}_{\mathrm{InCon}}$ $a, F, I, t, \alpha) \supset \sim G = \Lambda$.

TB2. $\vdash$ (G IntSubjAnlytcQuasiInt$^{1}_{\mathrm{InCon}}$ a, F, I, t, α . H IntSubj-AnlytcQuasiInt$^{1}_{\mathrm{InCon}}$ $a, F, I, t, \alpha) \supset G = H$.

TB3. $\vdash$ G IntSubjAnlytcQuasiInt$^{2}_{\mathrm{InCon}}$ $a, F, I, t, \alpha \supset \sim G = \Lambda$.

TB4. $\vdash$ (G IntSubjAnlytcQuasiInt$^{2}_{\mathrm{InCon}}$ a, F, I, t, α . H IntSubj-AnlytcQuasiInt$^{2}_{\mathrm{InCon}}$ $a, F, I, t, \alpha) \supset G = H$.

TB5. $\vdash$ (G IntSubjAnlytcQuasiInt$^{1}_{\mathrm{InCon}}$ a, F, I, t, α . H IntSubj-AnlytcQuasiInt$^{2}_{\mathrm{InCon}}$ a, F, I, t, α . $\sim$(Eb)(Ec)(Ed)(Eβ)(b S^{a}_{c} d . Anlytc b . $\beta > \alpha$. (X)(I $X \supset X$ Acpt b, F, t, β))) $\supset G = H$.

Suppose I consists of just the n persons $X_1, \ldots, X_n$. Let G_1 be X_1's subjective analytic quasi-intension of a of degree exactly α relative to F and t, G_2 X_2's, and so on. Then the intersubjective analytic quasi-intension of degree exactly α of a relative to F, I, and t is $(G_1 \cap G_2 \cap \cdots \cap G_n)$.

TB6. $\vdash$ (I X_1 . I X_2 . $\cdots$. I X_n . (X)(I $X \supset (X = X_1 \vee X = X_2 \vee \cdots \vee X = X_n)$) . G_1 SubjAnlytcQuasiInt$^{1}_{\mathrm{InCon}}$ a, X_1, F, t, α . $\cdots$. G_n SubjAnlytcQuasiInt$^{1}_{\mathrm{InCon}}$ a, X_n, F, t, α . H IntSubjAnlytcQuasiInt$^{1}_{\mathrm{InCon}}$ $a, F, I, t, \alpha) \supset H = (G_1 \cap G_2 \cap \cdots \cap G_n)$.

This theorem justifies the informal explanation above of intersubjective analytic quasi-intensions of degree exactly α as logical products of subjective analytic quasi-intensions of degree exactly α. An analogous theorem holds for intersubjective analytic quasi-intensions of degree α and upwards.

Let us consider again the primitive InCon 'a'. Let '--a--' again be an analytic sentence of F accepted at t by *all* members of the social group I to the degree exactly α. Any corresponding abstract '$x\ni$--x--' is then a member of the $\text{IntSubjAnlytcQuasiInt}^1_{\text{InCon}}$ of 'a'. If '--a--' is accepted by all members of I to some degree $\beta \geqq \alpha$ and to no degree $< \alpha$, then '$x\ni$--x--' is a member of the $\text{IntSubjAnlytcQuasiInt}^2_{\text{InCon}}$ of 'a' relative to F, I, t, and α.

Intertemporal analytic quasi-intensions are similar to intersubjective ones in the sense of being logical products of certain subjective analytic quasi-intensions, more specifically, of subjective analytic quasi-intensions of the *same person* at *different times*. Here also we have two kinds of intertemporal intensions, depending upon whether the subjective intensions involved are of degree exactly α or of degree α and upwards.

More precisely, we may let

'G $\text{IntTempAnlytcQuasiInt}^1_{\text{InCon}}$ a, X, F, α' abbreviate '(InCon a . $G = c\ni(\text{E}b)(\text{E}d)(\text{E}e)(c = (d \cap invep \cap e)$. $b\ \text{S}^a_d\ e$. Anlytc b . $(t)X$ Acpt b, F, t, α))'.

Also

'G $\text{IntTempAnlytcQuasiInt}^2_{\text{InCon}}$ a, X, F, α'

may be defined in a way analogous to the definition of '$\text{IntSubjAnlytcQuasiInt}^2_{\text{InCon}}$' and may be read '$G$ is the intertemporal analytic quasi-intension of a of degree α and upwards relative to X and F'.

If we wish to construe 'intertemporal' here as restricted to a given time interval, say to all times between and including t_1 and t_2, we may let

'G $\text{IntTempAnlytcQuasiInt}^3_{\text{InCon}}$ a, X, F, t_1, t_2, α' abbreviate '(InCon a . $G = c\ni(\text{E}b)(\text{E}d)(\text{E}e)(c = (d \cap invep \cap e)$. $b\ \text{S}^a_d\ e$. Anlytc b . $(t)(((t_1 \text{ B } t \vee t_1 = t) . (t \text{ B } t_2 \vee t = t_2)) \supset X$ Acpt b, F, t, α)) . $t_1 \text{B} t_2$)'.

The definiendum here may be read 'G is the intertemporal analytic quasi-intension of a of degree exactly α relative to X and F and restricted to the time interval t_1 to t_2'.

And similarly

$$\text{'}G \text{ IntTempAnlytcQuasiInt}^{4}_{\text{InCon}}\ a, X, F, t_1, t_2, \alpha\text{'}$$

may be defined to embody the notion that G is the intertemporal analytic quasi-intension of a of degree α *and upwards* relative to X and F and restricted to the time interval t_1 to t_2.

We need not ponder here the extent of the range of the variables for time. There is no need to assume that time stretches infinitely into the past or future. By 'for all times t' we need mean here merely 'for all times t under experimental consideration'.

Concerning these various notions, appropriate laws analogous to *TB1–TB6* may be proved.

We may also characterize intersubjective and intertemporal *veridical*, *synthetic*, and *theoremic* quasi-intensions for individual constants of degree exactly α or of degree α and upwards by inserting 'Tr b', 'Synthc b', or 'Thm b' in place of 'Anlytc b' in the definientia of these various definitions. These may all now be symbolized in obvious fashion, and the appropriate laws concerning them proved. Also intersubjective, etc., *contingent* quasi-intensions for individual constants of degree exactly α or of degree α and upwards may be introduced.

Also we have intersubjective and analytic, intertemporal, veridical, etc., quasi-intensions based on comprehension for one-place, two-place, etc., *predicate* constants. Again the definitions here are straightforward and may be omitted. The notions are to be symbolized in the familiar fashion.

Further types of intertemporal and intersubjective quasi-intensions for individual and predicate constants arise by considering again *L-equivalence*. The actual definitions here also are straightforward and the notions involved may be symbolized in the familiar way.

Here also it should be observed that each intersubjective or intertemporal quasi-intension is a virtual subclass of the corresponding objective quasi-intension of the same kind and for the same kind of constant.

C. SOCIETAL QUASI-INTENSIONS

More important perhaps than either intersubjective or intertemporal quasi-intensions are what we shall call *societal* quasi-intensions of degree exactly α or of degree α and upwards. Intersubjective quasi-intensions of a constant a are relative, as we have seen, to a virtual class F of sentences, to a virtual class I of persons, and to a time t, whereas intertemporal quasi-intensions of a are relative only to a person X and to a virtual class F of sentences. Societal quasi-intensions, now to be introduced, are relative neither to time nor to a person, but only to a virtual class F of sentences and to a social group I.

First we may let

'G SocAnlytcQuasiInt$^1_{\mathrm{InCon}}$ a, F, I, α' abbreviate '(InCon a . $\sim I = \Lambda$. $G = c \ni (\mathrm{E}b)(\mathrm{E}d)(\mathrm{E}e)(c = (d \cap \mathit{invep} \cap e)$. b S^a_d e . Anlytc b . $(X)(I\ X \supset (t)X$ Acpt $b, F, t, \alpha)))$'.

A virtual class of abstracts is a societal *analytic* quasi-intension of degree exactly α of an individual constant a relative to F and a nonnull social group I if and only if it consists of all abstracts determined by the appropriate analytic sentences of F *always* accepted by *all* members of I to degree exactly α.

Note that societal analytic quasi-intensions in this sense might also be called 'intertemporal analytic quasi-intensions of degree exactly α relative to the social group I', or equally well 'intersubjective analytic quasi-intensions of degree exactly α relative to all times under consideration'. But the term 'societal' seems to cover what is intended here more simply.

In a similar way societal analytic quasi-intensions of degree α *and upwards* may be introduced. We let, in now obvious fashion,

'G SocAnlytcQuasiInt$^2_{\mathrm{InCon}}$ a, F, I, α' abbreviate '(InCon a . $\sim I = \Lambda$. $(\mathrm{E}b)(\mathrm{E}c)(\mathrm{E}d)(b$ S^a_c d . Anlytc b . $(X)(I\ X \supset (t)X$ Acpt $b, F, t, \alpha))$. $\sim(\mathrm{E}b)(\mathrm{E}c)(\mathrm{E}d)(\mathrm{E}\beta)(b$ S^a_c d . Anlytc b . $\beta < \alpha$. $(X)(I\ X \supset (t)X$ Acpt $b, F, t, \beta))$. $G = c \ni (\mathrm{E}b)(\mathrm{E}d)(\mathrm{E}e)(\mathrm{E}\beta)(c = (d \cap \mathit{invep} \cap e)$. b S^a_d e . Anlytc b . $\beta \geqq \alpha$. $(X)(I\ X \supset (t)X$ Acpt $b, F, t, \beta)))$'.

Likewise societal analytic quasi-intensions restricted to certain time intervals may be introduced.

We may go on to characterize societal analytic quasi-intensions based on comprehension of degree exactly α or of degree α and upwards for predicate constants. Also we have societal veridical, synthetic, theoremic, and contingent quasi-intensions of degree exactly α or of degree α and upwards for individual constants, and societal veridical, synthetic, theoremic, and contingent quasi-intensions based on comprehension of degree α or of degree α and upwards for the various predicate constants. These notions may all be symbolized in familiar fashion. And finally we have also various types of societal quasi-intension based upon L-equivalence.

Apologies should perhaps be offered for this plethora of kinds of quasi-intension. Of course some of them are no doubt more important and interesting than others. We are not concerned here primarily with the application of these notions to actual usage. If we were, we should wish to pay more attention to some than others and to characterize them more fully. But from a theoretical point of view all of these notions should be defined and carefully distinguished from one another. Also here, as in (I, E–G), we have tried to use a reasonably suggestive terminology but perhaps have not altogether succeeded. The notions named are more important than the names themselves. These latter we may pay overtime and make them mean what we wish.

D. COINTENSIVENESS

We now turn to some relations of *cointensiveness*. In theories of meaning or semantical intension (see (V, D) below), such relations play an important role. So also here in the theory of subjective quasi-intensions, relations of cointensiveness are of especial interest. There are several different kinds to be distinguished, depending upon the kind of quasi-intensions implicitly involved. We shall consider only a few.

First, we may say that two individual constants a and b are *subjectively cointensive* relative to X, F, and t if and only if F contains the appropriate sentences and exhibits an RPP relative to X at t, and X accepts a suitable analytic sentence containing a to degree α if and only if he accepts to degree α the corresponding sentence containing b, for

all α. Of course this is somewhat vague, but we may make it precise by means of the following definition.

'a SubjCoInt$_{\text{InCon}}$ b, X, F, t' abbreviates '$((c)(e)(a')((\text{Anlytc } c \,.\, F\, c \,.\, c\, \text{S}_e^a\, a') \supset (\text{E}d)(F\, d \,.\, d\, \text{S}_e^b\, a')) \,.\, (d)(e)(a')((\text{Anlytc } d \,.\, F\, d \,.\, d\, \text{S}_e^b\, a') \supset (\text{E}c)(F\, c \,.\, c\, \text{S}_e^a\, a')) \,.\, (\text{E}c)(\text{E}d)(\text{E}e)(\text{E}a')(\text{E}\alpha)(\text{Anlytc } c \,.\, c\, \text{S}_e^a\, a' \,.\, d\, \text{S}_e^b\, a' \,.\, X \text{ Acpt } c, F, t, \alpha \,.\, X \text{ Acpt } d, F, t, \alpha) \,.\, (c)(d)(e)(a')(\alpha)((\text{Anlytc } c \,.\, c\, \text{S}_e^a\, a' \,.\, d\, \text{S}_e^b\, a' \,.\, F\, c \,.\, F\, d) \supset (X \text{ Acpt } c, F, t, \alpha \equiv X \text{ Acpt } d, F, t, \alpha)))$'.

The existence conditions on F here seem eminently reasonable, that it have as a member at least one analytic sentence containing a and one containing b accepted by X at the time to the same degree, that for every analytic sentence containing a in F there be a corresponding one containing b in F of the same form, to speak loosely, and that for every analytic sentence containing b in F there be one of the same form containing a instead. The InCon's a and b are then subjectively cointensive for X at t relative to the sentences of F provided he accepts an analytic sentence of F containing a at t to degree α if and only if he accepts the corresponding one containing b at t also to the same degree α, for all α. (The existence condition here may be weakened.)

This relation is reflexive, symmetric, and transitive in the appropriate senses and on suitable assumptions concerning F. Thus we have the following theorems.

TD1. $\vdash (\text{E}c)(\text{E}e)(\text{E}a')(\text{E}\alpha)(\text{Anlytc } c \,.\, c\, \text{S}_e^a\, a' \,.\, X \text{ Acpt } c, F, t, \alpha) \supset a$ SubjCoInt$_{\text{InCon}}$ a, X, F, t.

TD2. $\vdash a$ SubjCoInt$_{\text{InCon}}$ $b, X, F, t \supset b$ SubjCoInt$_{\text{InCon}}$ a, X, F, t.

TD3. $\vdash (a$ SubjCoInt$_{\text{InCon}}$ $b, X, F, t \,.\, b$ SubjCoInt$_{\text{InCon}}$ $c, X, F, t \,.\, (\text{E}d)(\text{E}e)(\text{E}a')(\text{E}b')(\text{E}\alpha)(\text{Anlytc } d \,.\, d\, \text{S}_e^a\, a' \,.\, b'\, \text{S}_e^c\, a' \,.\, X \text{ Acpt } d, F, t, \alpha \,.\, X \text{ Acpt } b', F, t, \alpha)) \supset a$ SubjCoInt$_{\text{InCon}}$ c, X, F, t.

Another kind of cointensiveness, as we might well expect, is *intersubjective* cointensiveness. We may let

'a IntSubjCoInt$_{\text{InCon}}$ b, F, I, t' abbreviate '$((c)(e)(a')((\text{Anlytc } c \,.\, F\, c \,.\, c\, \text{S}_e^a\, a') \supset (\text{E}d)(F\, d \,.\, d\, \text{S}_e^b\, a')) \,.\, (d)(e)(a')((\text{Anlytc } d \,.\, F\, d \,.\, d\, \text{S}_e^b\, a') \supset (\text{E}c)(F\, c \,.\, c\, \text{S}_e^a\, a')) \,.\, (\text{E}c)(\text{E}d)(\text{E}e)(\text{E}a')(\text{E}\alpha)(\text{Anlytc } c \,.\, c\, \text{S}_e^a\, a' \,.\, d\, \text{S}_e^b\, a' \,.\, (X)(I\, X \supset (X \text{ Acpt } c, F, t, \alpha \,.\, X \text{ Acpt } d, F, t, \alpha))) \,.\, {\sim} I = \Lambda \,.\, (c)(d)(e)(a')(\alpha)((\text{Anlytc } c \,.\, c\, \text{S}_e^a\, a' \,.\, d\, \text{S}_e^b\, a' \,.\, F\, c \,.\, F\, d) \supset (X)(I\, X \supset (X \text{ Acpt } c, F, t, \alpha \equiv X \text{ Acpt } d, F, t, \alpha))))$'.

In a similar way relations of *intertemporal* and *societal* cointensiveness may be introduced. The relations involved may be written in context as

$$\text{`}a \ \mathrm{IntTempCoInt}_{\mathrm{InCon}}\ b, X, F\text{'}$$

and

$$\text{`}a \ \mathrm{SocCoInt}_{\mathrm{InCon}}\ b, F, I\text{'}.$$

Clearly

TD4. $\vdash (a \ \mathrm{SubjCoInt}_{\mathrm{InCon}}\ b, X, F, t \,.\, G \ \mathrm{SubjAnlytcQuasiInt}^{1}_{\mathrm{InCon}}\ a, X, F, t, \alpha) \supset G \ \mathrm{SubjAnlytcQuasiInt}^{1}_{\mathrm{InCon}}\ b, X, F, t, \alpha.$

TD5. $\vdash (a \ \mathrm{IntSubjCoInt}_{\mathrm{InCon}}\ b, F, I, t \,.\, G \ \mathrm{IntSubjAnlytcQuasiInt}^{1}_{\mathrm{InCon}}\ a, F, I, t, \alpha) \supset G \ \mathrm{IntSubjAnlytcQuasiInt}^{1}_{\mathrm{InCon}}\ b, F, I, t, \alpha.$

The foregoing kinds of cointensiveness involve Anlytc. Corresponding kinds may be defined involving rather Synthc, Tr, or Thm. (Cf. again (V, D) below.)

These various cointensiveness relations take only individual constants as arguments. But there are also many cointensiveness relations appropriate to *predicate* constants as well. The definitions for these are closely patterned upon the foregoing and need not be given.

E. DEGREE OF RATIONAL BELIEF

The notion of a *quasi-proposition* we are familiar with from (I, F). The quasi-proposition corresponding to a given sentence *a* of *L* may be regarded as the virtual class of sentences L-equivalent with *a*. Propositions in this sense are relative to *L*, of course, and are linguistic in the sense of being virtual classes of linguistic objects. Later we shall wish to consider whether there are such things as nonlinguistic propositions, independent at least of any specific *L*. For these, however, a more powerful metalanguage will be needed.

The notion of *belief* is often characterized as a relation between a person and a proposition. No time factor is involved nor are beliefs subject to a quantitative measure. If we construe propositions as quasi-propositions for the moment, we may introduce a notion akin to that of belief within the pragmatical metalanguage here. Now that the notion of degree of acceptance is available, we have in effect a

quantitative measure for such a relation. And because beliefs frequently alter with the passage of time, the presence of a parameter for time seems desirable.

More precisely, we might say that person X believes the quasi-proposition F at time t to the degree exactly α if and only if F exhibits a RPP relative to X at t, F is the virtual class of sentences of L L-equivalent to some sentence a of L, and X accepts every member of F at time t to the degree α. Then we might let

'X Blvs F, t, α' abbreviate '$((\mathrm{E}a)(\mathrm{Sent}\ a \,.\, F = \mathrm{quasiProp}\ a \,.\, F\ \mathrm{RPP}\ X, t \,.\, (a)(F\,a \supset X\ \mathrm{Acpt}\ a, F, t, \alpha))$'.

However, the clause concerning 'Acpt' in the definiens here states in effect that the measure is defined over the whole virtual class F, an infinitely large class. Heretofore there has been no need to assume F infinite. Here we assume the measure defined over an infinite virtual class. The experimental basis of this definition is therefore somewhat doubtful. Hence it would be desirable to relativize the notion of belief to a possibly finite virtual subclass of F, and hence to presuppose the quantitative measure only for that virtual subclass. Thus we might let

'X Blvs F, G, t, α' abbreviate '$((\mathrm{E}a)(\mathrm{Sent}\ a \,.\, F = \mathrm{quasiProp}\ a \,.\, G\ \mathrm{RPP}\ X, t \,.\, G \subset F \,.\, (a)(G\,a \supset X\ \mathrm{Acpt}\ a, G, t, \alpha))$'.

The former definition is merely a special case of this.

We also may introduce the locution 'X believes the quasi-proposition F at time t to the degree α *and upwards*' in familiar fashion. Here also the measure may be presupposed for only a possibly finite virtual subclass of F.

These definitions are put forward tentatively merely as possible ways of explicating a relation (for the quasi-propositions of the systems L) closely akin at least to that of degree of rational belief. The explication of the notion of degree of rational belief itself requires a much deeper analysis than can be given here.[3]

[3] Cf. the author's "Toward an Extensional Logic of Belief," *The Journal of Philosophy* LIX (1962): 169–72, and "On Knowing, Believing, Thinking," *ibid.*: 586–600.

F. 'MORE ACCEPTABLE THAN'

In his book *Perceiving: A Philosophical Study*,[4] Chisholm has given an interesting preliminary and informal analysis of several pragmatical terms. Chisholm calls these terms 'epistemic'; or rather certain *uses* of the terms in question are called 'epistemic'. Such notions as *adequate evidence*, *acceptable*, *unreasonable*, *indifferent*, *certain*, *probable*, and *improbable* are often used, he points out, in appraising the epistemic or cognitive value or worth of sentences, propositions, beliefs, and the like. These words may perhaps be used in a nonepistemic sense also, but with such uses Chisholm is not concerned. Many of these terms appear to be pragmatical in the sense we are here discussing. For their analysis reference to a person seems required as well as to a time factor.

Chisholm's definitions go back ultimately to one undefined notion. This is, he tells us, the locution '*h is more worthy of X's belief* (*at t*) *than i*' or '*h is more acceptable* (*at t*) *to X than i*', where *h* and *i* are sentences or statements or even propositions. Although some space is devoted to the meaning of this primitive, no axioms or postulates are given characterizing it.

Let us attempt to define Chisholm's primitive—merely in passing and without going too deeply into the matter—using the notions of rational preference pattern, degree of rational belief, and so on, discussed above and in the preceding chapter. The notion to be defined here seems close enough surely to that of Chisholm to retain his terminology, and adequate perhaps for some of the uses to which his notion is put.

Precisely what sense of 'proposition' Chisholm allows his primitive locution to take on is not altogether clear. Let us construe propositions here for the present as sentences, but the definitions to be given can easily be extended to propositions in the sense of (I, F) or §E above if desired. Also here the discussion will be relative to a language-system, as throughout. Thus we aim to define the locution 'the sentence *a* (of *L*) is more acceptable to *X* at time *t* than the sentence *b* (of *L*)'.

When we have spoken of degree of acceptance above, the sentences concerned are all relative to a given virtual class of sentences under

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[4] (Ithaca: Cornell University Press, 1957).

experimental consideration at the time. Here likewise Chisholm's primitive locution will be relativized to a virtual class of sentences. This class should exhibit a rational preference pattern relative to the person at the time. Otherwise the notion of degree of acceptance is not defined. Further, this virtual class should exhibit a normal acceptance pattern with respect to the logical constants *tilde* and *vee*. This requirement seems needed to assure that these constants will behave in the normal way. If these various conditions are satisfied and person X accepts the sentence a relative to F at time t to a *greater* degree than he accepts the sentence b relative to F at time t, we then say that a is more acceptable to X than b at t.

Thus we may let

'a MA b, X, F, t' abbreviate '(F NAP *tilde*,X, t . F NAP *vee*, X, t . (Eα)(Eβ)(X Acpt a, F, t, α . X Acpt b, F, t, β . $\alpha > \beta$))'.

The element of being more *worthy* or more acceptable here is implicitly contained in the notion of an RPP involved in the definition of 'Acpt'. It might be objected that this is not sufficient. It is not altogether clear, however, precisely wherein the worthiness is supposed to lie. The relation here is relativized to X and a might be more acceptable to X at t than b but not to Y. The worthiness must thus be a function of X and is thus subjective. Of course if X is a paradigmatic user of L whose degrees of acceptance of a and b are their degrees of confirmation upon the total available evidence at the time, the worthiness is well-founded objectively. But this in general is not to be presupposed by the locution 'MA'. The worthiness here is fairly strong, being determined by the NAP- and the RPP-requirements. A weaker definition might be possible using only 'Prfr' and the NPR's of Chapter II. But it seems likely that the worthiness as based on such a definition would be too weak, being based merely on rankings rather than on quantitative patterns. Nonetheless this alternative should no doubt be explored. For the present we confine attention solely to 'MA' as defined above.

Perhaps also we should add in the definiens here further requirements concerning the normalcy of X's acceptances. For the present, we are concerned primarily with *tilde*, *vee*, and *dot*, however, and hence

only with normalcy with respect to these. And it is only with these that Chisholm is concerned anyhow.

A few elementary properties of this relation may be given as follows.

TF1. $\vdash a$ MA $b, X, F, t \supset {\sim}b$ MA a, X, F, t.

TF2. $\vdash (a$ MA $b, X, F, t \,.\, b$ MA $c, X, F, t) \supset a$ MA c, X, F, t.

TF3. $\vdash {\sim}a$ MA a, X, F, t.

TF4. $\vdash a$ MA $b, X, F, t \supset (F$ RPP $X, t \,.\, F\,a \,.\, F\,b)$.

Chisholm points out that the two locutions '*h* is more worthy of *X*'s belief than non-*h*' and 'non-*h* is more worthy of *X*'s belief than *h*' are contraries of each other, i.e., appropriate instances of these may both be false but both cannot be true. This observation is contained in effect in the following law.

TF5. $\vdash$ (*tilde a*) MA $a, X, F, t \supset {\sim}a$ MA (*tilde a*), X, F, t.

In view of the fact that *F* is to exhibit a NAP for *tilde* and *vee* where *a* MA *b*, *X*, *F*, *t*, we have also the following laws.

TF6. $\vdash (a$ MA $b, X, F, t \,.\, F$ (*tilde tilde a*) $.\ F$ (*tilde tilde b*)) $\supset$ (*tilde tilde a*) MA (*tilde tilde b*), X, F, t.

TF7. $\vdash (a$ MA $b, X, F, t \,.\, F$ (*tilde a*) $.\ F$ (*tilde b*)) $\supset$ (*tilde b*) MA (*tilde a*), X, F, t.

TF8. $\vdash ((a$ MA $b, X, F, t \vee c$ MA $b, X, F, t) \,.\, F\,a \,.\, F\,c \,.\, F$ (*a vee c*) $.\ F$ (*a dot c*)) $\supset$ (*a vee c*) MA b, X, F, t.

TF9. $\vdash (a$ MA (*b vee c*), $X, F, t \,.\, F\,b \,.\, F\,c \,.\, F$ (*a dot b*)) $\supset (a$ MA $b, X, F, t \,.\, a$ MA $c, X, F, t)$.

F. THE LOGIC OF EPISTEMIC TERMS

Let us suppose for the moment that Chisholm's primitive is available, either as above or in an improved form. Several further "epistemic" notions may then be defined, as Chisholm points out. Let us glance at some of these. In so doing, we by no means commit ourselves to Chisholm's epistemology, his doctrine of the synthetic *a priori*, or his "ethics of belief." We are interested here only in the logical interrelationships, in the logical geography, as it were, of these terms or notions, defined ultimately in terms of 'Prfr', 'Eq', and so on, the precise

interpretation of which is left open. But presumably suitable interpretations of these primitives can be given to yield part of Chisholm's theory. If so, the definitions of these notions will help to show that that theory involves, implicitly at least, notions of syntax, semantics, decision theory, and the theory of confirmation.

We may say that it is *unreasonable* for X to accept a of F at t if and only if (*tilde a*) is more acceptable to X at t than a, where of course both a and (*tilde a*) are in F. Thus

'Unrsnble X, a, F, t' may abbreviate '(*tilde a*) MA a, X, F, t'.

Chisholm points out that 'unreasonable' (or 'unreasonable to accept') here is roughly synonymous with 'absurd' or 'preposterous'. Of course it might be unreasonable or absurd for X to accept some a relative to F at t but not for some other person to do so. Also it might be unreasonable for X to accept a relative to F at some t but not for him to do so at some other time. And also it might be unreasonable for X to accept a at t relative to F but not for him to do so relative to some other virtual class of sentences.

Clearly,

TG1. $\vdash$ (Unrsnble X, a, F, t . F (*tilde tilde a*)) $\supset$ $\sim$Unrsnble X, (*tilde a*), F, t.

TG2. $\vdash$ (F a . F b . F (*a vee b*) . F (*a dot b*) . F (*tilde a*) . F (*tilde b*)) $\supset$ ((Unrsnble X, a, F, t . Unrsnble X, b, F, t) $\equiv$ Unrsnble X, (*a vee b*), F, t).

TG3. $\vdash$ (((Unrsnble X, a, F, t v Unrsnble X, b, F, t) . F (*a dot b*) . F (*a vee b*)) $\supset$ Unrsnble X, (*a dot b*), F, t.

But note that the converses of *TG1* and *TG3* do not in general hold.

Chisholm notes that whenever "it would be unreasonable for a man to accept a certain proposition, then he may be said to have *adequate evidence* for its contradictory." In accord with this we may let

'X Evid a, F, t' abbreviate 'Unrsnble X, (*tilde a*), F, t'.

Then we may say that a sentence a is *acceptable* to X relative to F at t if, roughly, a is in F and it is not unreasonable that he accepts a relative to F at t.

'Acptble *a*, *X*, *F*, *t*' abbreviates '(*F a* . *F* (*tilde a*) . *F* NAP *tilde*, *X*, *t* . *F* NAP *vee*, *X*, *t* . ~Unrsnble *X*, *a*, *F*, *t*)'.

The various clauses here in the definiens, that *F* exhibit NAP's with respect to *tilde* and *vee*, are required for obvious reasons. Without them, a sentence of *F* might fail to be acceptable to *X* at *t* merely because *F* failed to exhibit the appropriate pattern.

Chisholm notes that "if the contradictory of a proposition is not evident, then the proposition is acceptable." This principle is sharpened and embodied in *TG9* below.

Clearly we have the following theorems.

TG4. ⊢ (Unrsnble *X*, *a*, *F*, *t* . *F* (*tilde tilde a*)) ⊃ Acptble (*tilde a*), *X*, *F*, *t*.

By *TG1*.

TG5. ⊢ (*F a* . *F* (*tilde a*) . *F* (*tilde tilde a*) . *F* NAP *tilde*, *X*, *t* . *F* NAP *vee*, *X*, *t*) ⊃ (Acptble *a*, *X*, *F*, *t* v Acptble (*tilde a*), *X*, *F*, *t*).

By *TF5*.

TG6. ⊢ (*F a* . *F b* . *F* (*a vee b*) . *F* (*a dot b*) . *F* (*tilde a*) . *F* (*tilde b*)) ⊃ ((Acptble *a*, *X*, *F*, *t* v Acptble *b*, *X*, *F*, *t*) ≡ Acptble (*a vee b*), *X*, *F*, *t*).

By *TG2*.

TG7. ⊢ (Acptble (*a dot b*), *X*, *F*, *t* . *F a* . *F* (*tilde a*)) ⊃ Acptble *a*, *X*, *F*, *t*.

Also

TG8. ⊢ (*X* Evid *a*, *F*, *t* . *F a*) ⊃ Acptble *a*, *X*, *F*, *t*.

TG9. ⊢ (*F a* . *F* (*tilde a*) . *F* (*tilde tilde a*) . *F* (*tilde tilde tilde a*) . *F* NAP *tilde*, *X*, *t* . *F* NAP *vee*, *X*, *t* . ~*X* Evid (*tilde a*), *F*, *t*) ⊃ Acptble *a*, *X*, *F*, *t*.

A relation of indifference was introduced in Chapter II, it will be recalled, as a relation involving two sentences. That relation enabled us to express that a person is indifferent between a sentence *a* and a

sentence b at time t in the sense of not choosing one to the other or not preferring one to the other. A somewhat different indifference relation will now be defined involving the epistemic predicate 'Evid'. Thus

'Indiff a, X, F, t' abbreviates '($F\,a$. F (*tilde a*) . F (*tilde tilde a*) . F NAP *tilde*, X, t . F NAP *vee*, X, t . $\sim X$ Evid a, F, t . $\sim X$ Evid (*tilde a*), F, t)'.

The proposition to the effect that it will rain in London just a year from today is, Chisholm points out, epistemically indifferent for most persons. "If a proposition or hypothesis is indifferent, its contradictory is also indifferent. An indifferent proposition is thus one which is neither evident nor unreasonable." Thus

TG10. $\vdash$ (Indiff a, X, F, t . F (*tilde tilde tilde a*) . F (*tilde tilde tilde tilde a*)) $\supset$ Indiff (*tilde a*), X, F, t.

TG11. $\vdash$ (Indiff a, X, F, t . F(*tilde tilde tilde a*)) $\supset$ ($\sim X$ Evid a, F, t . $\sim$Unrsnble a, X, F, t).

TG12. $\vdash$ (Indiff a, X, F, t . X Acpt a, F, t, α . X Acpt (*tilde a*), F, t, β) $\supset \alpha = \beta$.

TG13. $\vdash$ ($F\,a$. F (*tilde a*) . F (*tilde tilde a*) . F NAP *tilde*, X, t . F NAP *vee*, X, t) $\supset$ (Indiff a, X, F, $t \equiv X$ Indiff a, (*tilde a*), t).

This theorem connects the two relations of indifference.

An *absolute skeptic*, Chisholm suggests, is one who holds that all propositions are epistemically indifferent. Relativizing this notion to a virtual class and a time, we may let

'X AbsSkept F, t' abbreviate '(F NAP *tilde*, X, t . F NAP *vee*, X, t . (a)($F\,a \supset$ Indiff a, X, F, t))'.

Note that none of the epistemic terms defined involves any notions of semantics. At some point, however, such notions might well be needed.

We need not tarry with the theory of epistemic terms any longer. But enough has been shown surely to illustrate the interesting interweaving of epistemology with syntax, confirmation theory, and quantitative pragmatics. We cannot claim to have put forward an adequately

worked-out epistemology without a fairly full development of these and allied areas. Perhaps some of the foregoing definitions are of value in helping to show that epistemology cannot satisfactorily be developed *in vacuo* and without careful attention to the logical and other notions being implicitly used or presupposed.

H. QUANTITATIVE PRAGMATICS BASED ONLY ON SYNTAX

The quantitative pragmatical metalanguage we have been considering is such as to contain both a syntax and a semantics as a part. Clearly a syntax must be presupposed because the pragmatical primitives 'Prfr' and 'Eq' take sentences of the object-language as arguments. A syntax is required to give appropriate syntactical properties of sentences. The question arises, however, as to whether we really need to presuppose also a semantics. Suppose we were to drop 'Den' as a primitive, the accompanying Rules of Denotation, as well as the (translation of the) object-language. The resulting quantitative pragmatics, although considerably weaker, actually would still suffice for most of the purposes for which it is intended.

To see this, we need merely note that very little of the preceding material would have to be sacrificed within a pragmatics presupposing only a syntax. Only occasionally has the semantical truth-concept been employed. The notion of analytic truth has figured fundamentally, but in view of the Adequacy Principle *TD8* of Chapter I, the syntactical correlate of being a sentence and a logical theorem may be used in its place. Thus if we replace 'Anlytc a' by 'Sent a . LogThm a' (with any other variable in place of 'a') throughout the various definitions and theorems of this chapter, we gain correlative definitions within a pragmatics presupposing only a syntax.

Thus only a few of the preceding definitions need be given up if semantics itself is given up. In particular, we should no longer have any *veridical* or *synthetic* quasi-intensions, but many of the purposes of the quantitative metalanguage may be served without these. Of course an *interpretation* for the object-language must be given up also, but the presence of the pragmatical notions helps to compensate for this, at least partially. Thus a good deal at least can be accomplished within a pragmatics based only on a syntax, but for other purposes

we might well wish to presuppose the fuller resources of a semantics.

I. QUASI-INTENSIONS AND ORDINARY LANGUAGE

The theory of subjective quasi-intensions has been explicitly relativized to a language-system *L*. The question arises as to whether, and if so how, this theory may be relativized to a full "ordinary" or natural language. To answer this properly would not be easy and would involve discussion of the precise relationship between logic (including language-systems) and natural language. Despite wide contemporary interest in this relationship, it seems that very little detailed experimental or theoretical work has actually been carried out to illumine it.

Let us reflect a little upon how the foregoing theory might be construed to provide a theory of subjective quasi-intensions for a natural language *NL*. In the first place we should have to assume that the *syntax* of *NL* involves an adaptation of at least the formal syntax of (I, C), developed in terms of concatenation. Surely concatenation or some allied notion plays a crucial role in the syntax of natural language, although this does not seem to be commonly recognized. Further we assume that *NL* itself contains the logical connectives 'not', 'or', etc., and that their behavior as applied at least to *some* declarative sentences obeys the usual logical laws. And similarly for quantifiers and identity. Further we need the theory of virtual classes in *NL*. Surely none of these assumptions is objectionable, as applicable at least to *some* sentences of *NL*.

We need assume, however, for the present no semantical notions. Their definition might involve difficulties. But semantical notions are not needed within a pragmatics presupposing only a syntax anyhow, as we have just noted. The question is, then, how much of the theory of subjective quasi-intensions can be preserved within a pragmatics of *NL* presupposing only a syntax?

Let us construe the virtual class *F* in the preceding definitions as consisting of some declarative sentences of *NL* together with some of their truth-functional components. The sentences of *F* may also contain quantifiers, the identity sign, and expressions for virtual classes. The members of *F* may thus be characterized without reference to the

general notion *declarative sentence in NL*. In fact it is not clear that we have a satisfactory definition of this latter at all.[5] This notion is a very involved one and at any event is not needed for present purposes.

We note now that none of the definitions in the *preceding* chapters involves any semantical concepts, if we exclude those of the notions of being a NPR or NAP relative to *LogThm*, *Thm*, *Tr*, or *Conf*. These definitions may be dropped altogether. But the other formal definitions may be retained unaltered upon the new interpretation for F and the concatenation symbol '$\frown$'. The material concerning preference as between nonlinguistic objects, in (II, G), involves a relation of designation or denotation and may be dropped or perhaps suitably reworked. But none of this dropped material plays any role in the theory of subjective quasi-intensions.

Turning now to the material of the present chapter, we notice that fundamental use is made of the notion Anlytc for L. It is mainly in order to have this notion available that we have needed a semantics for L at all. We might think that we could easily reconstrue 'Anlytc' to apply to *NL*. But if we have no satisfactory extensions of 'Sent' or 'Tr' to *NL*, it would surely be rash to assume one for 'Anlytc'. The *effect* of Anlytc may be achieved to some extent, however, by using (as in §H) the syntactical correlate of it as confined to F, the notion of being both a sentence of F and a logical theorem. If in place of 'Anlytc a' in the preceding definitions of this chapter we now put in a suitable expression for this notion, the resulting change is surprisingly slight. The usefulness of the various notions of this chapter for *NL* seems in no essential way impaired.

The foregoing comments are mere suggestions as to how this kind of pragmatical approach might be reworked to apply to a natural language. But to carry out such an application in detail would entail going into the syntactical and perhaps semantical structure of natural language rather deeply, and for this the help of the worker in empirical or structural linguistics would presumably be needed.

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[5] Cf., e.g., N. Chomsky, *Syntactic Structures* ('s Gravenhage: Mouton, 1957), and Z. Harris, *Methods in Structural Linguistics* (Chicago: University of Chicago Press, 1951).

V

INTENSIONS AND THE THEORY OF TYPES

Let us suppose now that the object-language is no longer a simple first-order L of the kind considered throughout, but rather involves a full theory of classes in the sense of the simplified theory of types. Within such an object-language many logical notions are definable that are not definable above. Some of these notions are of interest for the theory of intensions and quasi-intensions. Also the semantical metalanguage, as well as the quantitative pragmatical metalanguage, will contain the object-language as a part and hence also will be based on type theory. Let us attempt to formulate now a theory of both objective (semantical) and subjective (pragmatical) intensions and quasi-intensions within such a metalanguage.

It has already been remarked that genuine intensions are certain kinds of *nonlinguistic* entities. In fact, it seems quite clear that intensions should be so regarded if we are to be able to use them for certain important purposes. For example, we might wish to compare an expression in one language with one in another. We might wish to say that they do or do not have the *same* intension. For this reference to an intension outside language seems desirable, although perhaps not wholly indispensable.

In §A a brief sketch of *type theory* in the form to be employed is given, together with its *syntax* and *denotational semantics*. The so-called *objective veridical intensions* for *class constants* are introduced in §B and *objective analytic* and other intensions in §C. In §D objective intensions (of the various kinds) for *individual constants* are discussed, together with an alternative treatment of intensions for class constants. In §E, Frege's famous example concerning 'the morning star' and 'the evening star' is considered. *Objective quasi-intensions* are reintroduced in §F for the system based on types, and *genuine* subjective intensions in §G. Finally in §H we consider how the theory of genuine intensions may be developed for the *systems L* of Chapter I.

A. TYPE THEORY

The form of type theory we choose here is essentially that of Tarski.[1] An important feature of this formulation is that no relational variables are required. We shall need only variables for individuals and for classes of all types. But relations may be handled as classes of ordered pairs, triples, etc., using the devices of Wiener and Kuratowski.[2]

The *individual variables* range over a domain of objects of lowest type (type 1). The *class variables* of type 2 range over classes of these individuals. The class variables of type 3 range over classes of classes of these individuals, and so on. We let 'x', 'y', 'z' (possibly with primes or numerical subscripts) be the variables for individuals, 'M', 'N' (possibly with primes or numerical subscripts) the variables for classes of individuals, and 'κ' and 'λ' the variables for classes of classes of these individuals. Variables of still higher type may be introduced when needed. Most of our considerations here will be with terms of just these lower types, which can then easily be extended to higher types as well.

Universal quantifiers are admitted over each kind of variable. These are symbolized by '(x)', '(M)', '(κ)', etc. Existential quantifiers can be introduced by definition in the usual way.

[1] See *Principia Mathematica* and Tarski's *The Concept of Truth*, in *Logic, Semantics, Metamathematics*.

[2] N. Wiener, "A Simplification of the Logic of Relations," *Proceedings of the Cambridge Philosophical Society* 17 (1912–14): 387–90 and C. Kuratowski, "Sur la Notion de l'Ordre dans la Théorie des Ensembles," *Fundamenta Mathematicae* 2 (1921): 161–71.

The formal system here will be called *T*. We need not give a detailed formulation, the main features of which will be clear enough as we proceed.

We take as logical primitives '$\sim$' and 'v' as truth-functional connectives together with the universal quantifiers. Membership in a class is symbolized merely by the juxtaposition of terms. This also plays the role of a primitive, not as a logical primitive, but as a *nonlogical* one. The relation of object to class or of a class to a class of classes, is regarded here as a mathematical relation akin to the ε-relation of axiomatic set theory.

Formulae of the forms 'Mx', read 'the individual x is a member of the class M', and 'κM', read 'the class M is a member of the class of classes κ', for example, are to be atomic formulae.

It is desirable to have an operation of *abstraction* as part of the basic logic of *T*. We may let '$\hat{x}$---x---' stand for the class of all objects x such that ---x---, where '---x---' is a formula of *T* containing 'x' as a free variable. The use of the circumflex is an adaptation of that of *Principia Mathematica*. Expressions of the form '$\hat{x}$---x---' are called *abstracts* (of type 2). These abstracts are to be substituends, i.e., substitutable, for variables of type 2. In a similar way we have abstracts (of type 3) substitutable for variables of type 3, and so on.

Suppose we have an abstract '$\hat{x}$---x---' where '---x---' is a formula of *T* containing 'x' as its *only* free variable. '$\hat{x}$---x---' then stands for, more strictly *designates*, a given class. Abstracts of this kind we may call *defined* class constants.

We should observe the difference between the use of '^' here and that of '$\ni$' in the preceding chapters. Abstracts here are *substitutable for variables* and hence stand for *genuine classes* as values for variables. But the abstracts in the preceding chapters, formed by means of '$\ni$', do not stand for such entities. They give in effect a mere *manière de parler* and the entities for which they stand are thus merely *virtual*.

We may also include as primitives of *T* some nonlogical primitive individual or class constants of various types. The inclusion of these would be particularly appropriate were we to formalize within *T* some scientific or philosophical theory.

We have a choice as to the handling of identity. The identity sign '=' may be taken as a logical primitive of *T*, as a nonlogical primitive,

or it may be suitably defined. Identity is often taken as a basic relation of logic, it was so taken in (I, B) above, and therefore let us regard it as such here.

As logical axioms or rules we need the usual rules providing for the logic of the truth-functional connectives and quantifiers over all types of variables admitted. Also we need the following *Rule of Abstraction.*

Abst. $\vdash \hat{y}A\,x \equiv B$, where (etc., essentially as in (I, B)),

and similar Rules of Abstraction for variables of higher types.

Suppose we have just n distinct primitive individual constants, each of which is regarded as designating a unique entity. We should then have logical rules of identity stating that each of these entities is distinct from all others. And similarly for the primitive class constants of all types. And in addition we have the usual two basic Principles of Identity, analogous to *IdR1* and *IdR2*, for variables and constants of all types.

We might wish to make use of Russellian descriptions, at least for individuals, as a means of introducing defined individual constants. If so, we may suppose '℩' added as a primitive logical operator as applying to individual variables. Expressions of the form '(℩$x.A$)' are then regarded as terms handled in the manner of (I, B). We need then a special individual a*. Also we need an adaptation of the *Rule of Descriptions* as an additional logical rule. Descriptions for entities of higher type may also be introduced if desired, although no use will be made of them for the present.

We need as nonlogical axioms of T the so-called *Principles of Extensionality*, as follows.

Ext. $\vdash (x)(Mx \equiv Nx) \supset M = N$,

and similarly for variables of higher type. Also we need an Axiom of Infinity and Axioms of Choice, as well as other nonlogical axioms characterizing the nonlogical primitive individual and class constants.

We let 'V^1' abbreviate '$\hat{x}x = x$', 'V^2' '$\hat{M}M = M$', and so on. Similarly 'Λ^1' abbreviates '$\hat{x}{\sim}x = x$', and so on. '$M \subset N$' is to abbreviate '$(x)(Mx \supset Nx)$', and similarly for higher types. Also various Boolean operations may occasionally be needed. '$(M \cup N)$', for example, is to be short for '$\hat{x}(Mx \vee Nx)$', and so on.

The syntax of T we may suppose developed in the manner of (I, C), with such slight readjustments of detail as are needed. In particular we have '*circ*' now as the structural description of '^' in place of '*invep*' for '϶'. Then '$(ex \frown circ \frown ex \frown id \frown ex)$' may be regarded as the structural description of '$\hat{x}x = x$'. We may simply disregard here that we write the circumflex on top of the 'x' rather than as immediately following it.

We may let '$\text{Vbl}^1\ a$' express that a is a variable of first type, '$\text{Vbl}^2\ a$' that a is a variable of second type, and so on. Similarly we let 'InCon a' express that a is a primitive individual constant or a description containing no free variables (*defined* individual constant). We may let 'ClsCon a' express that a is a class constant of type 2, i.e., a primitive constant of type 2 or an abstract of the form '$\hat{\boldsymbol{x}}A$', where A is a formula of T containing no free variable other than the individual variable $\boldsymbol{x}$. Similarly we let 'ClsClsCon a' express that a is a constant of type 3 for classes of classes, i.e., a primitive constant of type 3 or an abstract of the form '$\hat{\boldsymbol{M}}A$' where A is a formula of T containing no free variable other than the class variable $\boldsymbol{M}$.

To provide the semantics of T let us presuppose a semantical metalanguage based on *designation*. Let 'Des' symbolize this relation. 'a Des x' states that the expression a designates the individual x, 'a Des M' states that a designates the class M, and so on. As *Rules of Designation* we have the following.

$DesR1_{\text{InCon}}$. $\vdash \boldsymbol{a}$ Des $\boldsymbol{x}$, where $\boldsymbol{x}$ is a primitive or defined individual constant and $\boldsymbol{a}$ is the structural description of that constant,

$DesR2_{\text{InCon}}$. $\vdash (a \text{ Des } x \,.\, a \text{ Des } y) \supset x = y$,

$DesR3_{\text{InCon}}$. $\vdash a \text{ Des } x \supset \text{InCon } a$,

$DesR1_{\text{ClsCon}}$. $\vdash \boldsymbol{a}$ Des $\boldsymbol{M}$, where $\boldsymbol{M}$ is a primitive or defined class constant (of type 2) and $\boldsymbol{a}$ is its structural description.

And so on, for class constants for all types, including corresponding forms of $DesR2_{\text{InCon}}$ and $DesR3_{\text{InCon}}$.

A suitable logical framework for the metalanguage for T based on designation is presupposed.

Upon the basis of 'Des' the semantical truth-concept for T may be defined as follows.

'Tr a' abbreviates '(Sent a . (Eb)(Vbl[1] b . ($b \frown circ \frown a$) Des $\hat{x}x = x$))',

where 'Sent a' expresses that a is a sentence of T. We can then prove the Adequacy Principle that

TA1. $\vdash$ Tr $a \equiv$ -----, where '-----' is any sentence of T and 'a' is taken as its structural description.

Similarly we let 'Anlytc a' express that a is an analytic or logical truth of T. This also may be defined on the basis of 'Des'.[3] An analytic truth of T is, roughly, one true wholly in virtue of its form, i.e., wholly in virtue of its structure as containing in certain ways the sentential connectives, the quantifiers, abstracts, the identity sign, and descriptions. We may also prove that this notion is adequate in the sense that the analytic or logical truths are just the logical theorems of T which are sentences. This is in effect a Completeness Principle, analogous to *TD8* of Chapter I.

Among the logical theorems or Anlytc's of T are included some sentences of the form

'$\hat{x}A$ a',

where A is, e.g., '$x = x$' or '$(M)(Mx \supset Mx)$'. But clearly no sentence of the form 'Ma' is an Anlytc, where M is any *primitive* nonlogical ClsCon.

We choose T as object-language, because as a matter of fact many important scientific and philosophical languages either have been or may easily be formalized within it. Philosophers in particular find languages such as T eminently "natural" and especially convenient to work with.

Also we have taken 'Des' here as a semantical primitive rather than 'Den' merely for technical convenience. But with only slight modifications in the foregoing we could have taken 'Den' instead.

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[3] The actual definition involves considerable technicality and hence may be omitted.

B. OBJECTIVE VERIDICAL INTENSIONS

Let us turn again to the theory of intensions, of genuine intensions now and not mere quasi-intensions. And first let us consider only objective intensions. But later, in §G, the requisite pragmatical relations may be reintroduced and a theory of subjective intensions developed as well.

There are many different types of genuine objective intensions to be distinguished. The first kinds we shall consider are due in part to Leonard.[4] These are intensions only for class constants. We shall consider individual constants again in §D.

Leonard speaks of objects and of their "characteristics", but it is perhaps not altogether clear how we are to construe 'characteristics'. Let us identify them with *classes* of objects in the sense of type theory. The main difference presumably between classes and characteristics is usually taken to be the condition of identity: two classes are identical if and only if their memberships coincide, whereas if two attributes or characteristics are identical then every object which has one has the other (but not necessarily conversely). If for the moment we construe characteristics as classes, we can still go on to formulate a theory of intensions, as we shall see.

First a few preliminary notions are needed. Following Leonard in essential respects, we may say that a class M is *common* to members of a class N if and only if every member of N is a member of M. Thus

'M Com N' abbreviates '$(x)(Nx \supset Mx)$' or '$N \subset M$'.

The definiendum here can also be read '*M comprehends N*'. (Cf. (I, G) above.) And a class M is *peculiar* to x where Mx.

'M Pclr x' is an alternative notation for 'Mx'.

A class M is *peculiar to members of N* or to *N simpliciter* if and only if N comprehends M, i.e., every member of M is a member of N.

'M Pclr N' abbreviates '$(x)(Mx \supset Nx)$' or '$M \subset N$'.

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[4] *Principles of Right Reason*, p. 234 ff.

A class of classes κ, on the other hand, is *jointly peculiar* to an object x, provided x is a member of *every* member of κ.

'κ JPclr x' abbreviates '$(M)(\kappa M \supset Mx)$'.

To say that κ is jointly peculiar to the individual x is to say essentially that x is a member of the *product* of κ (in the sense of *40.01, *Principia Mathematica.*) The product is definable as follows.

'p'κ' abbreviates '$\hat{x}(N)(\kappa N \supset Nx)$'.

The notion of joint peculiarity may be extended to a *class of objects*. We say that κ is *jointly peculiar* to the class of objects M if and only if every object to which κ is jointly peculiar is a member of M.

'κ JPclr M' abbreviates '$(x)(\kappa$ JPclr $x \supset Mx)$'.

Clearly

TB1. $\vdash \kappa$ JPclr $M \equiv M$ Com p'κ.

All of the foregoing definitions can be given within the object-language, and hence in (the translational part of) the metalanguage, and employ no notions of syntax or semantics. Several notions to follow, however, require the semantical notion of designation.

We introduce first what Leonard calls the *total contingent intension* of a class constant. To conform with the terminology above, we may call this rather the *objective veridical* intension. A class of classes κ will be the objective veridical intension of an expression a if and only if a is a class constant and there exists a class N such that (i) a Des N and (ii) κ is the class of all designated classes M common to N (or in which N is contained).

'κ ObjVerInt$_{\mathrm{ClsCon}}$ a' abbreviates '(ClsCon a . (EN)(a Des N . $\kappa = \hat{M}$((Eb)b Des M . M Com N)))'.

(The clause 'ClsCon a' here is not strictly needed in view of *DesR3*$_{\mathrm{ClsCon}}$.)

A class M is said to be *designated* provided (Eb)b Des M. We do not require that all classes be designated, but on the other hand only designated classes are of interest for present purposes. Intensions in general are to be *classes of designated classes*. We must be able to specify *in concreto* the members of an intension, and for this only designated classes seem suitable.

Note that for any class constant a there exists one and only one nonnull objective veridical intension of a.

TB2. $\vdash$ ClsCon $a \supset (\mathrm{E}\kappa)(\sim\kappa = \Lambda^2 \,.\, \kappa$ ObjVerInt$_{\mathrm{ClsCon}}$ $a \,.\, (\lambda)(\lambda$ ObjVerInt$_{\mathrm{ClsCon}}$ $a \supset \kappa = \lambda))$.

The proof utilizes the uniqueness law for designation *DesR2*$_{\mathrm{ClsCon}}$.

In view of the Adequacy Law for the truth-predicate we have the following equivalence.

TB3. $\vdash \kappa$ ObjVerInt$_{\mathrm{ClsCon}}$ $a \equiv$ (ClsCon $a \,.\, \kappa = \hat{M}(\mathrm{E}b)(b$ Des $M \,.$ Tr $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex))))$.

This law enables us to see the aptness of the terminology 'veridical' in speaking of objective veridical intensions.

Actually there are several kinds of intensions to be introduced, all distinguishable from each other by their membership. Leonard regards it as a defect of the older logic—as indeed it is of most modern theories of intension hitherto formulated— that they speak of *the* intension of a term, as though there were only one. The fact is that these theories have not discriminated the many different kinds of intension. The reason is probably twofold, as we have already in effect noted. No clear condition under which two intensions are the same is usually given. Also intensions are usually regarded as in some sense *sui generis* and hence how they involve or consist of or are generated out of other kinds of entities is not considered.

In general, an [objective] intension of a class constant is, for Leonard *mutatis mutandis*, any class of classes κ such that κ is jointly peculiar to the extension of that constant and its members are common to it. Thus

'κ ObjInt$_{\mathrm{ClsCon}}$ a' may abbreviate '(ClsCon $a \,.\, (\mathrm{E}M)(a$ Des $M \,.$ $(N)(\kappa N \supset N$ Com $M) \,.\, \kappa$ JPclr $M))$'.

Every particular kind of objective intension of a class constant must, then, be shown to be an objective intension in this sense.

Given any class constant a, there exists at least one class of classes κ such that κ is an objective intension of a. But uniqueness here of

course does not hold, except for ClsCon's designating the universal class.

TB4. $\vdash (\text{ClsCon}\ a\ .\ {\sim}a\ \text{Des}\ \text{V}^1) \supset (\text{E}\kappa)({\sim}\kappa = \Lambda^2\ .\ \kappa\ \text{ObjInt}_{\text{ClsCon}}\ a\ .\ {\sim}(\lambda)(\lambda\ \text{ObjInt}_{\text{ClsCon}}\ a \supset \kappa = \lambda))$.

TB5. $\vdash \kappa\ \text{ObjInt}_{\text{ClsCon}}\ a \equiv (\text{E}M)(a\ \text{Des}\ M\ .\ \text{p}‘\kappa = M)$.

An example will help to clarify these notions. Let T contain 'triangle' and other terms of three-dimensional Euclidean geometry as class constants and let the individuals (entities of lowest type) of T be the objects of that kind of geometry. The class of triangles, Leonard observes, is included in the classes of geometric objects which (i) lie in a plane, (ii) are bounded, (iii) have straight sides, and (iv) have three sides. Also the product of the class of classes determined by (i)–(iv) is a subclass of the class of triangles. In Euclidean geometry of three dimensions all triangles are members of the classes determined by (i)–(iv), and every member of the classes determined by (i)–(iv) is a Euclidean triangle. Thus the class of classes κ consisting of the classes determined by (i)–(iv) is an objective intension of the class constant 'triangle'. Every class in κ contains the class of triangles and p‘κ is a subclass of the class of triangles. But κ does not constitute an objective veridical intension of 'triangle'. There are other classes containing the class of triangles that are not included, e.g., the class of geometric entities having interior angles summing to 360° or less. Only if we add to κ *all* the (designated) classes truly containing the class of triangles, do we gain the objective veridical intension.

Leonard points out that the set of all characteristics common to the extension of a term is always jointly peculiar to the extension of that term. Clearly we have that

TB6. $\vdash \hat{M}(M\ \text{Com}\ N)\ \text{JPclr}\ N$.

The extension of a class term is here identified with the class itself, so that this formula seems adequately to express the suggested law. Its proof is immediate by noting that

$$\vdash (x)((M)(M\ \text{Com}\ N \supset Mx) \supset Nx).$$

Also if κ is the $\text{ObjVerInt}_{\text{ClsCon}}$ of a and a Des M, then κ is jointly

peculiar to M. And thus clearly if κ is the $\text{ObjVerInt}_{\text{ClsCon}}$ of a it is also an $\text{ObjInt}_{\text{ClsCon}}$ if a.

TB7. $\vdash (\kappa \ \text{ObjVerInt}_{\text{ClsCon}}\ a \,.\, a \ \text{Des}\ M) \supset \kappa \ \text{JPclr}\ M.$

TB8. $\vdash (\kappa \ \text{ObjVerInt}_{\text{ClsCon}}\ a \supset \kappa \ \text{ObjInt}_{\text{ClsCon}}\ a.$

TB9. $\vdash (\kappa \ \text{ObjInt}_{\text{ClsCon}}\ a \,.\, \lambda \ \text{ObjVerInt}_{\text{ClsCon}}\ a) \supset \kappa \subset \lambda.$

Let us suppose that V^1 consists of concrete objects and that 'R' and 'A' are available as ClsCon's designating respectively the class of rational beings and the class of animals. (Cf. the example at the end of (I, G) above.) We let

(1) 'M' be short for '$\hat{x}(\mathrm{R}x \,.\, \mathrm{A}x)$'.

The $\text{ObjVerInt}_{\text{ClsCon}}$ of 'M' consists then of R, A, and M, and *all* other designated classes of which M is a subclass. Any $\text{ObjInt}_{\text{ClsCon}}$ of 'M' will be a subclass of the $\text{ObjVerInt}_{\text{ClsCon}}$, as we know from *TB9*. Let κ contain merely R and A as members. κ is then clearly an $\text{ObjInt}_{\text{ClsCon}}$ of 'M' but not the $\text{ObjVerInt}_{\text{ClsCon}}$. κ is a subclass of the $\text{ObjVerInt}_{\text{ClsCon}}$ here but not conversely.

C. OBJECTIVE ANALYTIC INTENSIONS

Let us turn now to another kind of intension, which Leonard calls the *total strict* intension of a class term. For this the notion of *necessity* is needed. The intension of a class constant in this sense is to consist of all and only the necessary members of its objective veridical intension. But 'necessary' here can be construed in the sense of the predicate 'Anlytc' as above in §A, so that the intension in this sense may be called the objective *analytic* intension. Hence we may let

'$\kappa \ \text{ObjAnlytcInt}_{\text{ClsCon}}\ a$' abbreviate '$(\text{ClsCon}\ a \,.\, \kappa = \hat{M}(\mathrm{E}b)(b \ \text{Des}\ M \,.\, \text{Anlytc}\ (ex\ qu\ (a \frown ex\ hrsh\ b \frown ex))))$'.

Clearly every objective analytic intension is included in or is a subclass of the corresponding objective veridical intension, but not conversely.

TC1. $\vdash (\kappa \ \text{ObjAnlytcInt}_{\text{ClsCon}}\ a \,.\, \lambda \ \text{ObjVerInt}_{\text{ClsCon}}\ a) \supset \kappa \subset \lambda.$

Also

TC2. $\vdash$ ClsCon $a \supset (\mathrm{E}\kappa)(\sim\kappa = \Lambda^2 \,.\, \kappa$ ObjAnlytcInt$_{\mathrm{ClsCon}}\, a \,.\, (\lambda)(\lambda$ ObjAnlytcInt$_{\mathrm{ClsCon}}\, a \supset \kappa = \lambda))$.

TC3. $\vdash \kappa$ ObjAnlytcInt$_{\mathrm{ClsCon}}\, a \supset \kappa$ ObjInt$_{\mathrm{ClsCon}}\, a$.

A further kind of intension arises by replacing 'Anlytc' in the definiens here by 'Thm'. Such intensions may be called objective *theoremic* intensions of class constants. The relation involved may be symbolized by 'ObjThmInt$_{\mathrm{ClsCon}}$'. Clearly

TC4. $\vdash (\kappa$ ObjThmInt$_{\mathrm{ClsCon}}\, a \,.\, \lambda$ ObjVerInt$_{\mathrm{ClsCon}}\, a) \supset \kappa \subset \lambda$.

TC5. $\vdash (\kappa$ ObjThmInt$_{\mathrm{ClsCon}}\, a \,.\, \lambda$ ObjAnlytcInt$_{\mathrm{ClsCon}}\, a) \supset \lambda \subset \kappa$.

TC6. $\vdash \kappa$ ObjThmInt$_{\mathrm{ClsCon}}\, a \supset \kappa$ ObjInt$_{\mathrm{ClsCon}}\, a$.

To return to our example concerning *homo animal rationalis*. Clearly M, R, and A are members of the ObjThmInt$_{\mathrm{ClsCon}}$ of 'M'. Perhaps T ('two-legged') is also, but this depends upon the nonlogical axioms of the object-language. The membership of M, R, and A depends, on the other hand, only on the logical axioms.

Suppose now we replace 'Anlytc' or 'Thm' in the definientia of the two foregoing definitions by 'Synthc'. The resulting class of designated classes is also an intension, an objective *synthetic* intension. We may let 'κ ObjSynthcInt$_{\mathrm{ClsCon}}\, a$' symbolize that κ is the intension in this sense of the class constant a.

Some of the laws concerning objective synthetic intensions will require a suitable existence assumption (as indeed have some of the laws above concerning objective synthetic quasi-intensions). We have then that

TC7. $\vdash (\kappa$ ObjSynthcInt$_{\mathrm{ClsCon}}\, a \,.\, (\mathrm{E}N)(\mathrm{E}b)(b$ Des $N \,.\,$ Synthc $(ex\ qu\, (a \cap ex\ hrsh\ b \cap ex)) \,.\,$ Tr $(ex\ qu\, (b \cap ex\ hrsh\ a \cap ex)))) \supset \kappa$ ObjInt$_{\mathrm{ClsCon}}$ a.

TC8. $\vdash$ (ClsCon $a \,.\, (\mathrm{E}N)(\mathrm{E}b)(b$ Des $N \,.\,$ Synthc $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex)))) \supset (\mathrm{E}\kappa)(\sim\kappa = \Lambda^2 \,.\, \kappa$ ObjSynthcInt$_{\mathrm{ClsCon}}\, a \,.\, (\lambda)(\lambda$ ObjSynthcInt$_{\mathrm{ClsCon}}\, a \supset \kappa = \lambda))$.

Clearly if the class constants a and b have a common objective analytic or veridical or synthetic (on an hypothesis) or theoremic intension, then the classes designated by a and b are identical.

TC9. $\vdash (((\kappa\ \mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}\ a\ .\ \kappa\ \mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}\ b) \vee (\kappa\ \mathrm{ObjVerInt}_{\mathrm{ClsCon}}\ a\ .\ \kappa\ \mathrm{ObjVerInt}_{\mathrm{ClsCon}}\ b) \vee (\kappa\ \mathrm{ObjSynthcInt}_{\mathrm{ClsCon}}\ a\ .\ \kappa\ \mathrm{ObjSynthcInt}_{\mathrm{ClsCon}}\ b\ .\ (\mathrm{E}N)(\mathrm{E}c)(c\ \mathrm{Des}\ N\ .\ \mathrm{Synthc}\ (ex\ qu\ (a \cap ex\ hrsh\ c \cap ex))\ .\ \mathrm{Tr}\ (ex\ qu\ (c \cap ex\ hrsh\ a \cap ex)))\ .\ (\mathrm{E}N)(\mathrm{E}c)(c\ \mathrm{Des}\ N\ .\ \mathrm{Synthc}\ (ex\ qu\ (b \cap ex\ hrsh\ c \cap ex))\ .\ \mathrm{Tr}\ (ex\ qu\ (c \cap ex\ hrsh\ b \cap ex)))) \vee (\kappa\ \mathrm{ObjThmInt}_{\mathrm{ClsCon}}\ a\ .\ \kappa\ \mathrm{ObjThmInt}_{\mathrm{ClsCon}}\ b))\ .\ a\ \mathrm{Des}\ M\ .\ b\ \mathrm{Des}\ N) \supset M = N.$

But what amount to the converses of *TC9* need not hold. In general we should not expect that, given two identical classes, the objective analytic intensions of the class constants designating them are the same. This is an important property of objective analytic as well as of synthetic intensions, which is incorporated in the following *Principle of Intensionality.*

TC10. $\vdash \sim(M)(N)(a)(b)(\kappa)(\lambda)((M = N\ .\ a\ \mathrm{Des}\ M\ .\ b\ \mathrm{Des}\ N\ .\ ((\kappa\ \mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}\ a\ .\ \lambda\ \mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}\ b) \vee (\kappa\ \mathrm{ObjSynthcInt}_{\mathrm{ClsCon}}\ a\ .\ \lambda\ \mathrm{ObjSynthcInt}_{\mathrm{ClsCon}}\ b))) \supset \kappa = \lambda).$

But for objective veridical and theoremic intensions we do have that

TC11. $\vdash (M = N\ .\ a\ \mathrm{Des}\ M\ .\ b\ \mathrm{Des}\ N\ .\ ((\kappa\ \mathrm{ObjVerInt}_{\mathrm{ClsCon}}\ a\ .\ \lambda\ \mathrm{ObjVerInt}_{\mathrm{ClsCon}}\ b) \vee (\mathrm{Thm}\ (a \cap id \cap b)\ .\ \kappa\ \mathrm{ObjThmInt}_{\mathrm{ClsCon}}\ a\ .\ \lambda\ \mathrm{ObjThmInt}_{\mathrm{ClsCon}}\ b))) \supset \kappa = \lambda.$

Laws analogous to *TC9* and *TC10* hold for $\mathrm{ObjInt}_{\mathrm{ClsCon}}$ also.

To return to the example. Let a be 'M' and b be '$\hat{x}(\mathrm{T}x\ .\ \mathrm{A}x)$', i.e., '$(\mathrm{T} \cap \mathrm{A})$'. Clearly

$$\mathrm{M} = \hat{x}(\mathrm{T}x\ .\ \mathrm{A}x),$$

i.e., the class of men is the same as the class of two-legged animals. But the $\mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}$'s differ. The class R of rational beings is a member of the $\mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}$ of 'M' but not of that of '$(\mathrm{T} \cap \mathrm{A})$', because

$$`(x)((\mathrm{T} \cap \mathrm{A})x \supset \mathrm{R}x)'$$

is not an Anlytc where 'T', 'A', and 'R' are primitive ClsCon's.

It might be thought that *TC11* establishes that the objective veridical and theoremic intensions are in some sense not genuine intensions at all. But such a thought would be spurious in view of *TC6* and *TC7*.

In traditional theories of denotation and connotation there is supposed to obtain a so-called *inverse law*, that denotation and connotation vary inversely. This law seems never to have been stated too clearly. The following *Inverse Laws* seem to give reasonably close approximations to this traditional law.

TC12. $\vdash (\kappa$ ObjAnlytcInt$_{\text{ClsCon}}$ $a \,.\, \lambda$ ObjAnlytcInt$_{\text{ClsCon}}$ $b \,.$ Anlytc $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex))) \supset \lambda \subset \kappa.$

TC13. $\vdash (\kappa$ ObjVerInt$_{\text{ClsCon}}$ $a \,.\, \lambda$ ObjVerInt$_{\text{ClsCon}}$ $b \,.$ (Tr $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex))$ v (a Des $M \,.\, b$ Des $N \,.\, M \subset N))) \supset \lambda \subset \kappa.$

TC14. $\vdash ((\kappa$ ObjSynthcInt$_{\text{ClsCon}}$ $a \,.\, \lambda$ ObjSynthcInt$_{\text{ClsCon}}$ $b \,.$ Synthc $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex))$ v (κ ObjThmInt$_{\text{ClsCon}}$ $a \,.\, \lambda$ ObjThmInt$_{\text{ClsCon}}$ $b \,.$ Thm $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex)))) \supset \lambda \subset \kappa.$

Also we have some mixed Inverse Laws, typical of which are the following.

TC15. $\vdash (\kappa$ ObjSynthcInt$_{\text{ClsCon}}$ $a \,.\, \lambda$ ObjAnlytcInt$_{\text{ClsCon}}$ $b \,.$ Synthc $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex))) \supset \lambda \subset \kappa.$

TC16. $\vdash (\kappa$ ObjVerInt$_{\text{ClsCon}}$ $a \,.\, \lambda$ ObjSynthcInt$_{\text{ClsCon}}$ $b \,.$ Tr $(ex\ qu\ (a \cap ex\ hrsh\ b \cap ex))) \supset \lambda \subset \kappa.$

And so on.

To return again to the example. We note that M is analytically included in R and hence by *TC12* that the ObjAnlytcInt$_{\text{ClsCon}}$ of 'R' is a subclass of that of 'M'. This accords well with the traditional observation that the meaning of 'rational' is in some sense contained in that of 'man' (on the basis of the definition (1) of 'M' in §B above). Similarly M is synthetically contained in T, and hence the ObjSynthcInt$_{\text{ClsCon}}$ of 'T' is a subclass of that of 'M', in view of *TC14*.

The objective analytic, veridical, and theoremic intensions of the class constant 'Λ^1', i.e., $(ex \cap circ \cap tilde \cap ex \cap id \cap ex)$, designating the null class, are the class of all designated classes.

TC17. $\vdash (a = (ex \cap circ \cap tilde \cap ex \cap id \cap ex) \,.\, (\kappa$ ObjAnlytcInt$_{\text{ClsCon}}$ a v κ ObjVerInt$_{\text{ClsCon}}$ a v κ ObjThmInt$_{\text{ClsCon}}$ $a)) \supset \kappa = \hat{M}(\text{E}b)b$ Des $M.$

On the other hand, the objective synthetic intension of 'Λ^1' is itself null.

TC18. $\vdash (a = (ex \cap circ \cap tilde \cap ex \cap id \cap ex) \,.\, \kappa$ ObjSynthcInt$_{\text{ClsCon}}$ $a) \supset \kappa = \Lambda^2$.

The various objective intensions of 'V^1', designating the universal class, are merely the class whose only member is the universal class.

TC19. $\vdash (a = (ex \cap circ \cap ex \cap id \cap ex) \,.\, (\kappa$ ObjAnlytcInt$_{\text{ClsCon}}$ a v (κ ObjSynthcInt$_{\text{ClsCon}}$ a . (Eb)Synthc ($ex\ qu\ b \cap ex$)) v κ ObjVerInt$_{\text{ClsCon}}$ a v κ ObjThmInt$_{\text{ClsCon}}$ $a)) \supset \kappa = \hat{M}M = V^1$.

It is sometimes claimed that terms which have only a null designation or denotation have a universal intension or connotation. *TC17* may perhaps be regarded as giving a precise form to this claim for 'Λ^1'. But in view of *TC18* we note that this circumstance does not hold of its objective synthetic intension. Of course 'Λ^1' and 'V^1' are specific logical constants. In general for constants designating Λ^1 or V^1, the situation is different. For constants designating Λ^1 the following theorem obtains.

TC20. $\vdash (a$ Des Λ^1 . ((κ ObjAnlytcInt$_{\text{ClsCon}}$ a . Anlytc ($ex\ qu\ tilde \cap a \cap ex$)) v κ ObjVerInt$_{\text{ClsCon}}$ a v κ ObjThmInt$_{\text{ClsCon}}$ $a) \supset \kappa = \hat{M}$(E$b$)$b$ Des M.

Of the four types of objective intensions, the veridical are clearly the most inclusive, as we may observe from *TC1*, *TC4*, and the following.

TC21. $\vdash (\kappa$ ObjSynthcInt$_{\text{ClsCon}}$ a . λ ObjVerInt$_{\text{ClsCon}}$ $a) \supset \kappa \subset \lambda$.

Also, clearly the veridical intensions consist of just the analytic and synthetic ones together, but these latter need not be mutually exclusive.

TC22. $\vdash (\kappa$ ObjVerInt$_{\text{ClsCon}}$ a . λ ObjAnlytcInt$_{\text{ClsCon}}$ a . λ' ObjSynthcInt$_{\text{ClsCon}}$ $a) \supset \kappa = (\lambda \cup \lambda')$.

Of the various kinds of objective intensions the analytic intensions are perhaps the most important. The analytic intension of a class term seems to provide a reasonably close approximation to the *meaning* of that term. (Cf., however, (VI, H) below.) The other types of intension

are derivative and perhaps of interest primarily in comparison with or in contrast to analytic intensions.

We note that the objective veridical intension of a class term is infinitely large— assuming an infinity of individuals as provided by the Axiom of Infinity. The proof may be adapted here roughly as follows. Let M_1 be a member of the objective veridical intension of a predicate constant a, and let M_2 be any nonnull class not included in M_1. Then the logical sum $(M_1 \cup M_2)$ of the two classes M_1 and M_2 is also a member of the objective veridical intension of a and is distinct from M_1. If in fact there is an infinity of individuals, then given any M_1 in the objective veridical intension of a there is an infinity of classes of the kind M_2 distinct from M_1. But if there is only a finite number of individuals, this proof fails, because, given any M_1 there is at most a finite number of classes M_2 of the kind described.

It is not difficult to extend this theory to class constants of higher logical type. Also intensions of relations may easily be handled, relations being regarded here as classes of ordered couples. For constants of each type, their intensions are of next higher type, as is clear from the foregoing.

D. WHITEHEADIAN INTENSIONS

Leonard considers in passing how intensions of *individual* constants may be handled. Clearly individual constants have no intensions of the foregoing kinds, which are based on the logic of class inclusion. Indeed one may (as with Mill and Ryle for connotations) hold the view that individual constants have no intensions at all.[5]

One way of introducing intensions of individual constants, however, suggests itself as follows. Let us consider *unit* classes, i.e., classes with only one individual as a member. Let $\{x\}$ be the class whose only member is x. For a unit class constant a designating the class whose only member is x, the objective veridical intension of a might be regarded as κ where

$$\kappa = \hat{M}((\mathrm{E}b)b \text{ Des } M \,.\, \{x\} \subset M),$$

[5] See G. Ryle, "The Theory of Meaning," in *British Philosophy at Mid-Century*, ed. by C. A. Mace (London: George Allen and Unwin, 1957), pp. 239–64, and Mill's *A System of Logic*. Cf. Leonard, *op. cit.*, p. 341 ff.

in other words, where

$$\kappa = \hat{M}((\mathrm{E}b)b \text{ Des } M \,.\, Mx),$$

i.e., the class of all designated classes of which x is a member. Thus we may let

'κ ObjVerInt$_{\mathrm{InCon}}$ a' abbreviate '(InCon a . (Ex)(a Des x . $\kappa = \hat{M}(\mathrm{E}b)(b$ Des M . Mx)))'.

(Here also the phrase 'InCon a' is not strictly needed.) Clearly then

TD1. $\vdash$ InCon $a \supset (\mathrm{E}\kappa)(\sim\kappa = \Lambda^2$. κ ObjVerInt$_{\mathrm{InCon}}$ a . $(\lambda)(\lambda$ ObjVerInt$_{\mathrm{InCon}}$ $a \supset \kappa = \lambda$)).

TD2. $\vdash$ (κ ObjVerInt$_{\mathrm{InCon}}$ a . λ ObjVerInt$_{\mathrm{ClsCon}}$ b . a Des x . b Des M . $M = \{x\}) \supset \kappa = \lambda$.

TD3. $\vdash \kappa$ ObjVerInt$_{\mathrm{InCon}}$ $a \equiv$ (InCon a . $\kappa = \hat{M}(\mathrm{E}b)(b$ Des M . Tr $(b \cap a)$)).

Whitehead used on occasion to call the class $\hat{M}Mx$ the *essence* of x. Let us call the class $\hat{M}(\mathrm{E}b)(b$ Des M . Mx) the *designated essence* of x. It seems not unreasonable to identify the designated essence of x with the objective veridical intension of an individual constant designating x. Also in virtue of *TD3* an alternative definition in terms of 'Tr' could be given.

In an analogous way we may introduce objective analytic, synthetic, and theoremic intensions for individual constants. All of these are certain subclasses of designated Whiteheadian essences. They may be symbolized by 'ObjAnlytcInt$_{\mathrm{InCon}}$', etc., in familiar fashion.

Also we have certain further types of intensions for *class* constants, now regarded as designated essences. Such intensions may be said to be of *Whiteheadian* type. More particularly, we may speak of the *Whiteheadian analytic intension* of a class constant a as follows. We let 'μ' and 'ν' be variables now of *fourth* type. Then

'μ WhtdAnlytcInt$_{\mathrm{ClsCon}}$ a' may abbreviate '(ClsCon a . $\mu = \hat{\kappa}(\mathrm{E}b)(b$ Des κ . Anlytc $(b \cap a)$))'.[6]

Also we have Whiteheadian synthetic, veridical, and theoremic intensions of class constants in familiar fashion. The Whiteheadian

[6] Note that this definition gives essentially what is called the *absolute quasi-intension* of *a*, as defined on p. 88 of *Toward a Systematic Pragmatics*.

veridical intension of a, where a Des M, for example, consists of just those classes of designated classes of which M is truly a member. More specifically, we replace 'Anlytc' in the definiens above by 'Tr' to gain the definition of '$\mathrm{WhtdVerInt}_{\mathrm{ClsCon}}$', by 'Synthc' to gain that of '$\mathrm{WhtdSynthcInt}_{\mathrm{ClsCon}}$', and by 'Thm' to gain that of '$\mathrm{WhtdThmInt}_{\mathrm{ClsCon}}$'.

Note that the objective analytic, veridical, etc., intensions for *individual* constants are also of Whiteheadian type. We could also therefore symbolize the relations involved by '$\mathrm{WhtdAnlytcInt}_{\mathrm{InCon}}$', etc. We may have occasion in fact to refer to them in either way.

What in general is a Whiteheadian intension? In other words, what relation do we have here analogous to '$\mathrm{ObjInt}_{\mathrm{ClsCon}}$' above? In general, it would seem, we can say that a class of classes of type $(n + 2)$ is a Whiteheadian intension of a constant of type n provided there is at least one entity which that constant designates (i) which is a member of every member of that class of classes and (ii) such that there is no other entity a member of every member of that class of classes. For individual constants, we may let

'$\kappa\ \mathrm{WhtdInt}_{\mathrm{InCon}}\ a$' abbreviate '$(\mathrm{InCon}\ a\ .\ (\mathrm{E}x)(a\ \mathrm{Des}\ x\ .\ (M)(\kappa M \supset Mx)\ .\ \sim(\mathrm{E}y)(\sim y = x\ .\ (M)(\kappa M \supset My))))$'.

And for class constants of type 2,

'$\mu\ \mathrm{WhtdInt}_{\mathrm{ClsCon}}\ a$' abbreviates '$(\mathrm{ClsCon}\ a\ .\ (\mathrm{E}M)(a\ \mathrm{Des}\ M\ .\ (\kappa)(\mu\kappa \supset \kappa M)\ .\ \sim(\mathrm{E}N)(\sim N = M\ .\ (\kappa)(\mu\kappa \supset \kappa N))))$'.

And so on for class constants of higher type.

Clearly we have that

TD4. $\vdash (\kappa\ \mathrm{WhtdAnlytcInt}_{\mathrm{InCon}}\ a \vee \kappa\ \mathrm{WhtdVerInt}_{\mathrm{InCon}}\ a \vee \kappa\ \mathrm{WhtdThmInt}_{\mathrm{InCon}}\ a) \supset \kappa\ \mathrm{WhtdInt}_{\mathrm{InCon}}\ a.$

TD5. $\vdash (\kappa\ \mathrm{WhtdSynthcInt}_{\mathrm{InCon}}\ a\ .\ (\mathrm{E}M)(\mathrm{E}b)(\mathrm{E}x)(b\ \mathrm{Des}\ M\ .\ \mathrm{Synthc}\ (b \cap a)\ .\ a\ \mathrm{Des}\ x\ .\ (y)(My \supset x = y))) \supset \kappa\ \mathrm{WhtdInt}_{\mathrm{InCon}}\ a.$

And analogously for class constants, as well as for constants of higher logical type.

In general it might be thought that the analytic intension, in this Whiteheadian sense, of a class constant a designating a class M is closer to what we intuitively would wish to call the *meaning* of the

constant a designating M, than is the objective analytic intension in §B. (Cf. again (VI, H) below.) As constituents of the analytic intension of a we should wish to include perhaps not just the designated classes in which M is analytically contained, so to speak, as in §B, but all of the designated classes of classes of which M is analytically a *member*. The Leonard type of definition leaves out too much, it might be thought, being too closely patterned as it is on the logic of class inclusion. The analytic intension of a term is determined by the whole bundle of the analytic properties of its extension, it could be maintained, and not just by the analytic classes in which that extension is included.

Technically speaking, the Whiteheadian kind of definition has the advantage of being uniformly the same for constants of all types, including individual constants. The Leonardian definitions in §B, on the other hand, have the advantage, if such it be, of treating intensions as only one type higher than their terms. On the basis of the Whiteheadian definitions, intensions are *two* types higher than their terms. Where the whole ontology of types is at one's disposal, the advantage of the Leonardian definitions would seem slight. On the other hand, if one's ontology is to be restricted to only a few lower types, this advantage might well be significant. But basically there is no quarrel here. We should recognize rather that there are these two distinct families of intensions for class constants. Intensions of each kind may be useful for some purposes.

The membership and hence the number of parts of an intension are often infinitely large, so that as a rule the whole intension of a constant cannot be fully specified. But this should not be regarded as a disadvantage or shortcoming of the present kind of theory. On the contrary, the intension (of a given kind) of a term is often infinitely complex and the best we can do is thus to specify a *good deal* of it. This good deal must of course include the items which are of especial interest to us or which are important in the given pragmatic context.

We note the following interconnection between the notion of being a Whiteheadian intension and products of classes of classes.

TD6. $\vdash \kappa \; \mathrm{WhtdInt}_{\mathrm{InCon}} \, a \equiv (\mathrm{InCon} \; a \, . \, (\mathrm{E}x)(a \; \mathrm{Des} \; x \, . \, \mathrm{p}`\kappa = \{x\}))$.

TD7. $\vdash \mu \; \mathrm{WhtdInt}_{\mathrm{ClsCon}} \, a \equiv (\mathrm{ClsCon} \; a \, . \, (\mathrm{E}M)(a \; \mathrm{Des} \; M \, . \, \mathrm{p}`\mu = \{M\}))$.

Also if two InCon's have the same Whiteheadian intension, those constants designate the same individual.

TD8. $\vdash (\kappa \text{ WhtdInt}_{\text{InCon}}\, a \,.\, \kappa \text{ WhtdInt}_{\text{InCon}}\, b \,.\, a \text{ Des } x \,.\, b \text{ Des } y) \supset x = y.$

And similarly for ClsCon's.

Let us now turn to some relations of *cointensiveness* between constants. As a matter of fact, there turns out to be a good variety of relations here, most of which have not been distinguished in the literature heretofore.

Two individual constants may be said to be *analytically cointensive* provided there exists a κ such that κ is the objective analytic intension of both.

'$a \text{ AnlytcCoInt}_{\text{InCon}}\, b$' abbreviates '$(\text{E}\kappa)(\kappa \text{ ObjAnlytcInt}_{\text{InCon}}\, a \,.\, \kappa \text{ ObjAnlytcInt}_{\text{InCon}}\, b)$'.

And similarly for constants of higher type.

Clearly L-equivalent constants are analytically cointensive and conversely.

TD9. $\vdash a \text{ LEquiv}_{\text{InCon}}\, b \equiv a \text{ AnlytcCoInt}_{\text{InCon}}\, b.$

A few further laws concerning analytic cointensiveness are given in (VI, B).

Equivalent constants a and b are those such that $(a \frown id \frown b)$ is Tr, where '*id*' is the structural description of '='. Equivalent InCon's are just those which have an $\text{ObjVerInt}_{\text{InCon}}$ in common.

TD10. $\vdash (\text{InCon } a \,.\, \text{InCon } b) \supset ((\text{Tr } (a \frown id \frown b) \equiv (\text{E}\kappa)(\kappa \text{ ObjVerInt}_{\text{InCon}}\, a \,.\, \kappa \text{ ObjVerInt}_{\text{InCon}}\, b)).$

And similarly for constants of higher type.

What about constants having a common theoremic intension? Clearly

TD11. $\vdash (\text{InCon } a \,.\, \text{InCon } b) \supset (\text{Thm } (a \frown id \frown b) \equiv (\text{E}\kappa)(\kappa \text{ ObjThmInt}_{\text{InCon}}\, a \,.\, \kappa \text{ ObjThmInt}_{\text{InCon}}\, b)).$

And similarly for constants of higher types.

We could go on to consider other kinds of cointensiveness, as between constants having a common objective synthetic intension and

as between constants one of which has one kind of intension and the other another. The character of such relations depends rather intimately upon the primitives as well as upon the nonlogical axioms of the object-language. Some of these notions might be of interest for specific object-languages. But we need not consider them further here, where we merely note them as theoretical possibilities.

Still further kinds of cointensiveness emerge by taking into account the Whiteheadian intensions of ClsCon's. Here also laws analogous to *TD8*, *TD9*, and *TD10* obtain. Here likewise further kinds, including mixed kinds, may be defined, some of which might be of interest for special systems. Some of them reduce to kinds previously introduced or mentioned.

E. 'THE MORNING STAR' AND 'THE EVENING STAR'

Let us consider Frege's famous example concerning 'the morning star' and 'the evening star' on the basis of the present theory. According to Frege these two phrases designate the same object, the planet Venus, but have different meanings—they have the same *Bedeutung* but different *Sinne*.[7]

Within *T* let '$\mathrm{S}x$' read 'x is a star', '$\mathrm{M}x$', 'x shines in the morning', and '$\mathrm{E}x$' 'x shines in the evening', where 'S', 'M', and 'E' are primitive ClsCon's. We disregard here all irrelevant astronomical matters, all matters concerned with the flow of time, precise characterizations of 'morning', 'evening', etc., as is customary, whether justly or not. (We may presuppose all this adequately handled in a suitable *T*.) We suppose also that 'ms' and 'es' are distinct (defined) individual constants, defined in such a way that

$$\text{`ms} = (\imath x \,.\, (\mathrm{M}x \,.\, \mathrm{S}x))\text{'}$$

and

$$\text{`es} = (\imath x \,.\, (\mathrm{E}x \,.\, \mathrm{S}x))\text{'}$$

are both analytic sentences. The notation '$(\imath x \,.\, \text{--}x\text{--})$' is that of Russell's theory of singular descriptions, it will be recalled, which are included here primitively in *T*. The objective veridical intensions of 'ms' and 'es'

[7] See "Über Sinn und Bedeutung," in *Philosophical Writings*, pp. 56–78.

are identical, because as a matter of astronomical fact ms = es. The objective analytic intensions, however, differ because we can find a member of the objective analytic intension of 'ms' not a member of that of 'es'. Clearly '$\hat{x}x$ = ms ms' is analytic, whereas '$\hat{x}x$ = ms es' is not, and hence $\hat{x}x$ = ms is a member of the objective analytic intension of 'ms' but not of that of 'es'. Let κ $\mathrm{ObjAnlytcInt}_{\mathrm{InCon}}$ 'ms' and λ $\mathrm{ObjAnlytcInt}_{\mathrm{InCon}}$ 'es'. Then, as we have just noted, $\hat{x}x$ = ms is a member of κ but not of λ.

Let us consider another example, similar to that in (I, G). Let T contain as primitive ClsCon's 'H' ('husband(s) of Xantippe'), 'GP' ('Greek philosophers'), and 'D' ('drinkers of hemlock'). Let

'Soc' be short for '$(\gamma x\,.\,(\mathrm{GP}x\,.\,\mathrm{D}x))$'.

Clearly then

'$\mathrm{Soc} = (\gamma x\,.\,(\mathrm{GP}x\,.\,\mathrm{D}x))$'

is analytic in T, and

'$\mathrm{Soc} = (\gamma x\,.\,\mathrm{H}x)$'

is true in T but not analytically so. Hence $\hat{y}y = (\gamma x\,.\,(\mathrm{GP}x\,.\,\mathrm{D}x))$ is a member of the $\mathrm{ObjAnlytcInt}_{\mathrm{InCon}}$ of 'Soc' but not of that of '$(\gamma x\,.\,\mathrm{H}x)$'.

F. OBJECTIVE QUASI-INTENSIONS AND TYPES

Let us consider again the theory of objective *quasi*-intensions of Chapter I, in order to reconstrue it here on the basis of the theory of types.

We must presuppose of course the syntactical and semantical adjustments suggested in §A.

First we may introduce objective quasi-intensions based on L-equivalence for individual or class constants of T and achieve quantification over them, essentially as (I, E). And similarly for quasi-propositions based on L-equivalence, essentially as in (I, F).

Objective analytic quasi-intensions for individual constants of T may be introduced as follows.

'F $\mathrm{ObjAnlytcQuasiInt}_{\mathrm{InCon}}$ a' abbreviates '$(\mathrm{InCon}\ a\,.\,F = c\ni(\mathrm{E}b)(\mathrm{E}d)(\mathrm{E}e)(c = (d\frown circ\frown e)\,.\,\mathrm{Vbl}^1\,d\,.\,b\ \mathrm{S}^a_d\,e\,.\,\mathrm{Anlytc}\ b))$'.

(The 'S^a_d' here is to be presumed suitably defined within the syntax of T where a and d are terms *of the same type*. Alternatively we could

require here that e be a formula containing d as its only free variable and let $b = (d \frown circ \frown e \frown a)$.) This definition is similar to that of (I, G). Objective veridical, synthetic, and theoremic quasi-intensions for individual constants may be introduced in an analogous way. Also we may introduce objective analytic, etc., quasi-intensions for class constants of all types as based upon comprehension, essentially as in (I, G).

Finally, objective quasi-intensions of Whiteheadian type may be introduced for class constants of all types. For example, we may let

'F WhtdAnlytcQuasiInt$_{\mathrm{ClsCon}}$ a' abbreviate '(ClsCon a . $F = c \ni$ (Eκ)(c Des κ . Anlytc ($c \frown a$)))'.

(The quantifier '(Eκ)' and the clause 'c Des κ' serve merely to fix c as a class constant of one type higher than that of a.) And similarly for Whiteheadian synthetic, veridical, and theoremic quasi-intensions.

Now that genuine intensions are available for T, it might seem that the quasi-intensions are of little interest. In a way quasi-intensions are a substitute for genuine intensions and mirror or represent them, as it were, where genuine intensions are not available. Nonetheless, the quasi-intensions are forthcoming here in the metalanguage for T merely by definition, as we see, and hence we have them willy-nilly. (Cf. (VI, C) below.)

G. SUBJECTIVE INTENSIONS AND TYPES

Let us turn again to the theory of *subjective* quasi-intensions, reconstruing it now on the basis of the theory of types. Here we may develop also a theory of genuine subjective intensions, which we may then interrelate with the genuine objective intensions introduced above.

The new primitives 'Prfr' and 'Eq', together with 'B', may now be introduced within the semantical metalanguage for T based on designation. The various formal definitions of Chapters II and III may now be presupposed as suitably adapted to T. These for the most part may be given without difficulty. (We assume, if needed, a suitable characterization of degree of confirmation for the sentences of T.)

The subjective quasi-intensions discussed in Chapter IV are all *expressional*, of course, in the sense of consisting of virtual classes of expressions. The definitions there may also be adapted here, but with

some slight changes due to the different handling of abstraction. These may easily be supplied. For example, we let

'G SubjAnlytcQuasiInt$^1_{\mathrm{InCon}}$ a, X, F, t, α' abbreviate '(InCon a . $G = c\ni(\mathrm{E}b)(\mathrm{E}d)(\mathrm{E}e)(c = (d{\frown}circ{\frown}e)$. Vbl^1 d . b S^a_d e . Anlytc b . X Acpt b, F, t, α))'.

And similarly for the other kinds of subjective quasi-intensions, *mutatis mutandis*.

Now that a full theory of classes is available, however, we can introduce *genuine* subjective intensions as certain kinds of *classes* of nonlinguistic objects.

For subjective analytic intensions of the first kind for individual constants, we may let

'κ SubjAnlytcInt$^1_{\mathrm{InCon}}$ a, X, F, t, α' abbreviate '(InCon a . $\kappa = \hat{M}(\mathrm{E}b)(\mathrm{E}c)(\mathrm{E}d)(\mathrm{E}e)(c$ Des M . Vbl^1 d . $c = (d{\frown}circ{\frown}e)$. b S^a_d e . Anlytc b . X Acpt b, F, t, α))'.

A class of classes κ is a SubjAnlytcInt$^1_{\mathrm{InCon}}$ of degree α of a primitive or defined individual constant a relative to X, F, and t if (roughly) κ is the class of all designated classes M such that the individual x designated by a is analytically a member of M, and X accepts at t to degree α the sentence of F that x is a member of M.

In a similar fashion we may introduce subjective veridical, synthetic, theoremic, and contingent intensions of degree exactly α for individual constants of T. And for each of these kinds, we also have related subjective intensions of degree α and upwards. We then go on to the intersubjective, intertemporal, and societal analytic, synthetic, veridical, theoremic, and contingent intensions of degree exactly α and of degree α and upwards for individual constants. The definitions of all of these notions may easily be given, being similar to the definitions of the corresponding quasi-intensions of Chapter IV.

For each type here for which there is a corresponding genuine intension, the subjective intension is a suitable subclass of the corresponding genuine intension. For example, we have that

TG1. $\vdash$ (κ SubjAnlytcInt$^1_{\mathrm{InCon}}$ a, X, F, t, α . λ ObjAnlytcInt$_{\mathrm{InCon}}$ a) $\supset \kappa \subset \lambda$.

All of these notions of subjective intension may also be introduced for class constants. The definitions are straightforward and present no difficulty.

Note that we do not introduce subjective intensions corresponding to the objective quasi-intensions based on L-equivalence. The classes involved would all be merely the class designated by the given constant. Genuine subjective intensions based on L-equivalence would thus be rather trivial and uninteresting.

It should also be observed that there are no objective contingent intensions or quasi-intensions of any kind. All contingent intensions are subjective, requiring for their definition a clause concerning acceptance.

H. INTENSIONS AND THE SYSTEMS *L*

Note that for the systems *L* of Chapter I we have no genuine intensions at all. The metalanguages for *L* based on denotation contain no class variables and hence no expressions for genuine intensions of any kind.

We can, however, formulate a theory of genuine intensions for *L* by employing as a metalanguage a suitable adaptation of a part of the metalanguage for *T*. We shall not formulate such a metalanguage in detail but merely note some of its features.

To simplify let *L* contain for the moment only *one-place* primitive predicate constants. These may then be identified in effect with the primitive *class* constants of *T*. The primitive individual constants of *L* likewise may be identified with those of *T*. The defined individual constants of *L* may be identified with a certain subset of the defined individual constants of *T* (with just those containing no variables of type higher than 1 and no constants of type higher than 2). And similarly the defined predicate constants of *L* may be identified with a certain subset of the defined class constants of *T*. *L* then may be regarded as a subsystem of *T* in a certain sense. Hence of course the metalanguage for *T* may be used as a metalanguage for *L* also. Actually we should not need all of this metalanguage but only a very small part. In fact we should need only the variables of the first four types.

To construct the theory of genuine intensions for such a simplified

L within such a metalanguage, we could proceed precisely as above. Objective intensions for the primitive and defined individual and predicate constants of all kinds may be introduced by suitable adaptations of the foregoing definitions.

If *L* contains relational constants as primitive, a suitable extension must of course be made in the metalanguage. For this we may add variables of still higher types or else introduce relational variables as well as variables over classes of relations, and so on, as needed. Here we may if desired regard relations as entities *sui generis* rather than handle them by means of the devices of Wiener and Kuratowski.

We see then that the semantics of *L* may be formulated in these two quite different metalanguages, one much more powerful than the other. For the purposes of just a denotational semantics for *L* we need only the narrower metalanguage. But for its intensional semantics the more powerful metalanguage must be employed.

VI

ADEQUACY, COMMITMENT, AND MEANING

In this chapter the discussion of subjective and objective intensions and quasi-intensions is continued. More specifically we are concerned here with the sense or senses if any in which the foregoing definitions may be said to be *adequate*. We then go on to the discussion of various kinds of *commitment*, which in turn presuppose the theory of intensions. Finally a somewhat new notion of a *pragmatical system* is put forward.

In §A the *total classification* of the various kinds of intension introduced is reviewed. Several requirements of adequacy are suggested in §B. In §C the advantages of genuine over quasi-intensions are considered, especially with regard to the *interlinguistic comparison* of intensions. In §D and §E several different kinds of commitment are considered under the two broad heads of *existential* and *intensional* commitment. In §F the possibility of developing a theory of *genuine propositions*, as over and against the quasi-propositions of Chapter I, is considered. In §G syntactical, semantical, and pragmatical systems are compared and contrasted. §H is concerned with how 'meaning' might be introduced as a technical term. Finally

in §I some further remarks concerning intensions and meaning in ordinary language are put forward.

A. THE TOTAL CLASSIFICATION

The plethora of intensions and quasi-intensions we have met with in the preceding chapters has not been artificially contrived or thought up. As a matter of fact, there simply is this enormity of kinds or types of intension waiting as it were to be characterized. Let us review briefly now the various types, gaining therewith a synoptic view of the total classification.

First, for the systems L, we have quasi-intensions of both objective and subjective type. Of objective (semantical) quasi-intensions for individual constants, we have just five kinds: objective L-equivalent quasi-intensions (the relation involved being symbolized by '$\mathrm{ObjLEquivQuasiInt}_{\mathrm{InCon}}$'), objective analytic quasi-intensions ('$\mathrm{ObjAnlytcQuasiInt}_{\mathrm{InCon}}$'), and objective veridical, synthetic, and theoremic ones ('$\mathrm{ObjVerQuasiInt}_{\mathrm{InCon}}$', '$\mathrm{ObjSynthcQuasiInt}_{\mathrm{InCon}}$', and '$\mathrm{ObjThmQuasiInt}_{\mathrm{InCon}}$'). Of quasi-intensions for one-place predicate constants of L, there are five corresponding types ('$\mathrm{ObjLEquivQuasiInt}_{\mathrm{PredConOne}}$' etc.). Four of these quasi-intensions are of the comprehensional type. And similarly for PredConTwo's, etc. In addition to these we have quasi-propositions as based on L-equivalence ('QuasiProp').

Of subjective (or pragmatical) quasi-intensions for individual constants of L, we have just twelve types as follows:

$\mathrm{SubjLEquivQuasiInt}^1_{\mathrm{InCon}}$,	$\mathrm{SubjLEquivQuasiInt}^2_{\mathrm{InCon}}$,
$\mathrm{SubjAnlytcQuasiInt}^1_{\mathrm{InCon}}$,	$\mathrm{SubjAnlytcQuasiInt}^2_{\mathrm{InCon}}$,
$\mathrm{SubjVerQuasiInt}^1_{\mathrm{InCon}}$,	$\mathrm{SubjVerQuasiInt}^2_{\mathrm{InCon}}$,
$\mathrm{SubjSynthcQuasiInt}^1_{\mathrm{InCon}}$,	$\mathrm{SubjSynthcQuasiInt}^2_{\mathrm{InCon}}$,
$\mathrm{SubjThmQuasiInt}^1_{\mathrm{InCon}}$,	$\mathrm{SubjThmQuasiInt}^2_{\mathrm{InCon}}$,
$\mathrm{SubjContQuasiInt}^1_{\mathrm{InCon}}$,	$\mathrm{SubjContQuasiInt}^2_{\mathrm{InCon}}$.

For one-place predicate constants, we have a similar classification:

$\mathrm{SubjLEquivQuasiInt}^1_{\mathrm{PredConOne}}$,	Etc.,
Etc.	

And likewise for PredConTwo's, etc.

We have also twelve corresponding types of intersubjective quasi-intensions for individual constants:

$\mathrm{IntSubjLEquivQuasiInt}^{1}_{\mathrm{InCon}}$, Etc.
Etc.;

twenty-four corresponding types of intertemporal quasi-intensions for individual constants:

$\mathrm{IntTempLEquivQuasiInt}^{1}_{\mathrm{InCon}}$, $\mathrm{IntTempLEquivQuasiInt}^{2}_{\mathrm{InCon}}$,
Etc., Etc.;
$\mathrm{IntTempLEquivQuasiInt}^{3}_{\mathrm{InCon}}$, $\mathrm{IntTempLEquivQuasiInt}^{4}_{\mathrm{InCon}}$,
Etc., Etc.;

and twelve types of societal quasi-intensions:

$\mathrm{SocLEquivQuasiInt}^{1}_{\mathrm{InCon}}$, Etc.,
Etc.

A similar classification obtains for PredConOne's, PredConTwo's, etc. All in all for *L* we have distinguished sixty-five types of quasi-intension for InCon's, sixty-five for PredConOne's, and so on, plus the quasi-propositions.

All of the objective quasi-intensions for *L* are definable within the semantical metalanguage based on denotation. The subjective, inter-subjective, etc., types of quasi-intension are definable in the pragmatical metalanguage containing that semantical metalanguage as a part. But suitable syntactical correlates of some of the subjective and other quasi-intensions are definable within the pragmatical metalanguage presupposing only a syntax.

Turning now to the systems *T*, we have a classification of objective quasi-intensions (similar to that for *L*) for individual constants and for class constants of all types, namely, five kinds for each. And here also we have quasi-propositions. In addition to these, we have quasi-intensions of the Whiteheadian kind for class constants:

$\mathrm{WhtdAnlytcQuasiInt}_{\mathrm{ClsCon}}$,
$\mathrm{WhtdVerQuasiInt}_{\mathrm{ClsCon}}$,
$\mathrm{WhtdSynthcQuasiInt}_{\mathrm{ClsCon}}$,
$\mathrm{WhtdThmQuasiInt}_{\mathrm{ClsCon}}$;

and similarly for constants of each higher type.

Of subjective (or pragmatical) quasi-intensions for individual constants of T we have the twelve kinds just as in L, and similarly for class constants of each type. Also we have the twelve kinds of intersubjective, the twenty-four kinds of intertemporal, and the twelve kinds of societal quasi-intensions for individual constants; and similarly for class constants of each type.

Finally for T we also have genuine objective and subjective intensions in addition to the quasi-intensions. First we have the four kinds of objective intensions of the Leonardian variety for class constants:

$$\mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}},$$
$$\mathrm{ObjVerInt}_{\mathrm{ClsCon}},$$
$$\mathrm{ObjSynthcInt}_{\mathrm{ClsCon}},$$
$$\mathrm{ObjThmInt}_{\mathrm{ClsCon}};$$

and similarly for class constants of higher types. For the individual constants of T we have the four varieties of objective intensions of Whiteheadian kind:

$$\mathrm{ObjAnlytcInt}_{\mathrm{InCon}},$$
Etc.

And in addition, the four kinds of Whiteheadian intension for class constants:

$$\mathrm{WhtdAnlytcInt}_{\mathrm{ClsCon}},$$
Etc.;

and similarly for class constants of higher types.

For some types of subjective quasi-intension there is a corresponding type of genuine subjective intension, as we have noted in §G of the preceding chapter.

For the individual constants of T, we have the various kinds of genuine subjective, intersubjective, intertemporal, societal analytic, veridical, synthetic, theoremic, or contingent intensions of degree exactly α or of degree α and upwards. And similarly for constants of higher type.

The objective (including Whiteheadian) intensions and quasi-intensions for T are definable within the semantical metalanguage for T based on designation. The subjective and other kinds of intensions and quasi-intensions of course are definable only within the quantitative pragmatics for T. Here also syntactical correlates of some of the

subjective and other quasi-intensions for *T* are definable within a pragmatical metalanguage for *T* presupposing only a syntax.

Turning again to the systems *L*, we noted in (V, H) that the theory of genuine intensions for *L* requires a more powerful metalanguage based on type theory. But within such a metalanguage all the various kinds of objective and Whiteheadian intensions may be defined for the primitive and defined constants of *L*. Also if this richer metalanguage for *L* were extended to include a quantitative pragmatics, various kinds of genuine subjective intensions could also be introduced.

It is not claimed that this total classification exhausts all the important types of intension or quasi-intension for *L* and *T*. There may be others which should be added. The ones considered here arise naturally from the preceding notions of semantics and thus recommend themselves as presumably the most significant. Also they seem to have some kinship however remote with traditional notions. Because so little work has been done in classifying and characterizing within modern semantics and pragmatics the great variety of intensions and quasi-intensions, this list is perhaps acceptable as a first attempt.

B. TOWARD A NOTION OF ADEQUACY

Let us attempt now to characterize the sense or senses, if any, in which the various foregoing definitions of intension or quasi-intension might be said to be *adequate*.

In general a criterion of adequacy for a given definition, or for a notion or expression introduced by a definition, is a statement or a series (conjunction) of statements of the metalanguage—in this case in the metametalanguage—to the effect that such and such a desirable condition or conditions holds or hold of the notion or expression so introduced. In the case of the truth-predicate for *T*, we have the principle *TA1* of (V, A) assuring a kind of adequacy. In the case of 'Anlytc' we have the Completeness Principle mentioned there. And indeed, for every notion we introduce, it is reasonable to ask: What is the (or a) condition of adequacy for this notion? To give a clear and unambiguous answer is often very difficult. But without at least an attempt at an answer, it is often not easy to grasp the purport of a definition or to judge whether or to what extent the definition is successful.

We shall not attempt here to give an exact definition of 'adequacy' for any of the definitions of intension given. Instead we shall approximate such a definition by listing certain general requirements which the above definitions do in fact satisfy, and which, it would seem, all definitions of intension should satisfy. Some of these will be quite exact, but some of them will perforce be less so. No doubt some at least of these items should be taken account of in any satisfactory notion of adequacy for a definition of intension.

First we should wish to require that the metalanguage in which the theory of intensions is developed be suitably restricted. By this is meant roughly that the metalanguage be reasonably limited as to primitive modes of expression and axiomatic structure. The importance of suitably restricted metalanguages for semantics has been discussed in some detail elsewhere.[1] Of course the metalanguage cannot be too restricted. It must contain (i) the usual syntax, (ii) a designation or denotation (or other) relation with appropriate properties, (iii) an adequate truth-concept, and (iv) an adequate concept for analytic (logical) truth. But nothing further is needed, i.e., no primitives or axioms over and above those needed for (i)–(iv). Of course each of the items (i)–(iv) needs further elaboration, but they are intended here in essentially the senses of (I, C–D) and (V, A). If the metalanguage were to contain more than this, it might contain too much. To employ too much is in effect not to analyze deeply enough. Economy in this rough sense and depth of analysis usually go hand in hand.

Next, genuine intensions should be nonlinguistic entities in order to make possible interlinguistic comparison. The most natural and direct way of comparing the intension of a term in one language with that of a term in another is no doubt to have a suitable nonlinguistic entity available as an intermediary. Under 'nonlinguistic' here are included entities which are neither expressions nor classes (or virtual classes) of such of any kind. It is only by getting outside of language altogether, it would seem, that we can arrive at intensions at all. This point has frequently been recognized in the literature. (Cf. §C below.)

Thirdly, intensions should be grounded fundamentally upon the

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[1] Cf. *Truth and Denotation*, p. 263 ff., *The Notion of Analytic Truth*, p. 91 ff., and *Toward a Systematic Pragmatics*, p. 89 ff.

notions of truth, analytic truth, synthetic truth, or theoremhood. Without such notions available it is not clear that we could develop a satisfactory theory of intensions at all. The very use of the adjectives 'analytic', 'veridical', etc., in speaking of the various kinds of intension is to suggest this very fundamental interconnection. Some writers have treated intensions independent of these notions, but this is to disregard or at any event gloss over important interconnections.

Another requirement for intensions concerns analytic cointensiveness or L-equivalence. Suppose a and b are InCon's, and '--*a*--' is the structural description of a sentence of L containing a. Let the structural description '--*b*--' differ from '--*a*--' only in containing '*b*' wherever '*a*' occurs in '--*a*--'. Then clearly we have that

TB1. $\vdash a\ \mathrm{AnlytcCoInt}_{\mathrm{InCon}}\ b \supset \mathrm{Anlytc}\,(\text{--}a\text{--}\ \textit{tripbar}\ \text{--}b\text{--})$,

and similarly for analytic cointensiveness for class constants of each type. Hence

TB2. $\vdash a\ \mathrm{AnlytcCoInt}_{\mathrm{InCon}}\ b \supset (\kappa\ \mathrm{ObjAnlytcInt}_{\mathrm{InCon}}\ a \equiv \kappa\ \mathrm{ObjAnlytcInt}_{\mathrm{InCon}}\ b)$.

Thus analytically cointensive individual constants have the same objective analytic intension. Similarly for class constants. Also we have

TB3. $\vdash a\ \mathrm{AnlytcCoInt}_{\mathrm{InCon}}\ b \supset ((\kappa\ \mathrm{ObjVerInt}_{\mathrm{InCon}}\ a \equiv \kappa\ \mathrm{ObjVerInt}_{\mathrm{InCon}}\ b) \cdot (\kappa\ \mathrm{ObjThmInt}_{\mathrm{InCon}}\ a \equiv \kappa\ \mathrm{ObjThmInt}_{\mathrm{InCon}}\ b))$,

and similarly for class constants. These various laws of cointensiveness clearly should hold, although there is no analogous law for objective synthetic intensions.

Another requirement for a genuine intension of a class constant is essentially that of Leonard. We should wish to require, namely, that an intension of a given constant is a class of classes κ (i) whose product is contained in and (ii) all of whose members comprehend, the class designated by that constant. That this requirement gives a reasonable criterion for being an intension for class constants has in effect been shown implicitly at least above in (V, B and C). And all the Leonardian intensions of class constants in fact satisfy it, as is shown by *TB8*, *TC3*, *TC6*, and *TC7* of Chapter V. And similarly for class constants of higher types.

For intensions of individual constants and for intensions of Whiteheadian type for class constants, we need, of course, another criterion. Here we should wish to require, for an individual constant a, that a class of classes κ is an intension of a provided the individual designated by a is a member of *all* members of κ and there is no other individual of which this is true. The analogous condition should hold of class constants. That these criteria yield reasonable kinds of intensions of Whiteheadian type has in effect been shown above. And all the intensions of Whiteheadian type introduced are such as to satisfy it. (Cf. *TD4* and *TD5* of Chapter V and their analogues for ClsCon's.)

Associated with each nonlogical constant of the language, there should be one and only one intension of the given kind. This requirement of *univocality* is commonly assumed to hold of intensions. It is reasonable, therefore, to adopt it here.

Another very important requirement for any kind of intension is that, given any constant, we should be able explicitly to *specify* its intension of that kind. By 'specify' here one means concretely to enumerate or otherwise enunciate the various members or components or constituents of that intension. This *requirement of explicit specifiability*, as we may call it, presupposes in fact that intensions do have specifiable constituents. Without constituents in some sense or other, it is not clear that there are such things as intensions at all. It will not do merely to say, as some logical theorists have, that the *individual concept* Walter Scott is the intension of the individual constant 'Walter Scott'. To be told that there are such things as individual concepts or intensions *sui generis* in some sense, without being shown how we can analyze or decompose them into their components, does not seem very helpful or illuminating.

Any theory of intension should be able to give concrete meaning to the phrase '*part of*' in the sense of 'part of' used when we say that such and such is a part of the meaning or intension of such and such a term or phrase. This requirement, of having a clear notion of part of, is, of course, akin to the requirement of explicit specifiability. When we say that such and such is a part of the intension of such and such an expression, we might be thought to be specifying one of the members or components of that intension. Because intensions are certain kinds of classes, however, we must be careful to distinguish between being a

member of (in the sense of class-membership) an intension and being a *proper subclass* of it. The requirement of explicit specifiability is concerned with membership. The present requirement is concerned with being a part of in the sense of being a proper subclass of.

Still another requirement, for some kinds of intension at least, is, roughly, that although two classes or objects may be the same it need not follow that the intension (of some one kind) of the terms designating them are. This characteristic of intensions has frequently been noted in the literature, particularly by Frege and Russell. Clearly a class *M* may be the same as a class *N*, but it need not follow that the objective analytic intension of a constant designating *M* is the same as that of a constant designating *N*. That this requirement is satisfied for objective analytic and synthetic intensions of class constants is shown by the Principle of Intensionality *TC10*(V). We have also observed in *TC11*(V), that this requirement fails for objective veridical and (on an hypothesis) for objective theoremic intensions of class constants.

It is frequently said that although a term may have no actual reference, it always has a sense, i.e., that it may designate no actual thing but still have an intension. Frege in particular notes this as an important characteristic of the terms of a natural language although it is an undesirable one, he says, for terms of a "logically perfect" language. Because of the technical way in which descriptions for individuals are handled here, however, all descriptive phrases designate. (If either the uniqueness or existence condition is not satisfied, the description designates the object a*.) Likewise all primitive and defined class constants designate, if no nonnull class, then, at least the null class. Suppose for the moment that 'round' ('R') and 'square' ('S') are class constants of *T*. Then the abstract

$$`\hat{x}(\mathrm{R}x \,.\, \mathrm{S}x)'$$

designates only the null class. But of course this abstract has a nonnull objective intension, for example, just as does any individual description designating a*.

We note also that although the designata of two class constants may both be the null class, the objective analytic intensions of the two constants may be quite different. Let 'G' ('golden') and 'M'

('mountain') be class constants of *T*. Then, using also the class constant of the preceding paragraph,

$$\hat{x}(\mathrm{G}x \,.\, \mathrm{M}x) = \hat{x}(\mathrm{R}x \,.\, \mathrm{S}x) = \Lambda^1.$$

Yet the objective analytic intensions of these two constants are not null and are very different from each other.

To summarize. We have suggested the following requirements for adequate definitions of the various kinds of genuine intensions: (1) the metalanguage in which they are given must be suitably restricted, (2) intensions must be nonlinguistic entities, (3) their definition should depend fundamentally upon the notions of truth, analytic truth, synthetic truth, and theoremhood, (4) suitable laws of cointensiveness should obtain, (5) the Leonard condition should hold for intensions of class constants and the Whiteheadian one for intensions of individual constants and for intensions of Whiteheadian type of class constants, (6) each nonlogical constant should have a unique intension of the given kind, (7) we should have a clear notion of being a component or constituent member of an intension, (8) we should have a clear notion of being a part of or of being included in an intension, (9) suitable so-called principles of intensionality should obtain, and (10) all terms should have nonnull analytic intensions, but perhaps only degenerate designata.

It is not claimed that these various requirements are the only ones needed. Perhaps there are further ones. The items mentioned, however, seem rather central to the theory of intensions, at least as conceived here, and most of them have some intuitive justification. Some of them are more or less traditional, such as (2), (4), and (6). Also some are closely linked conceptually with others, and thus they are not intended to be independent necessarily of each other.

Let us reflect now upon adequacy for *quasi*-intensions. In what sense or senses can a definition of a kind of objective quasi-intension be said to be adequate? First, of course, quasi-intensions must be handled as virtual classes of expressions. This is involved in the use of the prefix 'quasi' throughout. Also, roughly stated, the members of the given quasi-intension of the constant must stand for a member of the corresponding genuine intension. The phrase 'stand for' here is deliberately vague. For the systems *T* it can mean 'designate', but

not for the systems L because expressions for virtual classes of objects do not (within the metalanguage for L) designate. But if we formulate L within the more powerful metalanguage suggested in (V, H), it should then obtain in that metalanguage that any member of a given quasi-intension of a constant of L designates a member of the corresponding genuine intension.

We may also ask in what sense if any the definitions for the various types of subjective intension might be said to be adequate. But no very clear answer seems to be forthcoming. Each subjective intension of some one kind (for which there is a corresponding kind of objective intension) is a *subclass* of the corresponding objective intension of the same kind, as illustrated, for example, by *TG1*(V). Hence we might require that a subjective intension (other than the subjective *contingent* intension) be a subclass of the corresponding genuine intension. Also, clearly subjective intensions should depend upon X and t (and derivatively upon F and α). The use of 'subjective' throughout is intended to suggest this dependence. Given X, t, F, and α, the subjective intension of a given kind is thus uniquely determined as a certain subclass of the corresponding objective intension of the same kind. This requirement, or rather these two requirements, seem to give all that is needed by way of adequacy for definitions of subjective intension.

C. INTERLINGUISTIC COMPARISON

It remains to consider the advantage or advantages which genuine intensions have over quasi-intensions, whether subjective or objective.

Suppose we wish to *translate* a constant belonging to one language-system L into another L'. The relation of *being a translation of* is itself a relation within a *comparative semantical metalanguage* containing both L and L' in some way as parts. Let us suppose that L and L' are given a uniform calligraphy, and suppose the theory of intensions is formulated within a semantical metalanguage containing logical types in such a way that within it we can compare L and L' systematically. Let a be some individual constant in L and a' one in L'. Then a has a certain objective analytic intension definable in the comparative metalanguage, as does a'. Suppose the objective analytic intension of a is in fact the *same* class of classes as the objective analytic intension of a'. We should then regard a as *translatable* into a' and conversely. By

referring to this class of classes as a nonlinguistic intermediary we characterize the exact relationship between a and a'. If no such intermediary were available, the notion of being a translation of would remain in doubt. Of course some third language-system L'' might be invoked and some individual constant a'' within it, and a of L and a' of L' compared somehow by reference to a'' of L''. But this would presuppose use of the very relation of being a translation of which we are seeking to characterize.

For class constants analogous circumstances obtain. Also a characterization must be given of the notion that a *logical* constant of L is translatable into one of L', and similarly for quantifiers. We should then define translation as between a sentence of L and one of L', perhaps in terms of constituent intensions in common.

We do not attempt to define the notion of translation rigorously, presupposing as it does a comparative semantical metalanguage. Such a metalanguage remains to be formulated, although some of its features are clear enough from the foregoing. Within such a metalanguage, we should hope to be able to formulate an exact comparative semantics in which different systems could be compared systematically not only with respect to their syntax and denotational or designational semantics, but with respect to their intensional semantics as well. We should then have a suitable foundation for comparative pragmatics.

D. EXISTENTIAL COMMITMENT

Here and there above Quine's criterion of ontological or ontic commitment has been mentioned. According to this criterion, roughly speaking, we are ontically committed by the sentences of a language L, or by the language L as a whole, to the entities over which the variables of the language range. This criterion has been more or less presupposed throughout.

In a recent symposium paper, Church has put forward an "emendation" to Quine's criterion, which is in effect three-fold.[2] "Especially

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[2] A. Church, "Ontological Commitment," *Journal of Philosophy* 55 (1958): 1008–14. Some of the material of this section is borrowed from the author's "On Church's Notion of Ontological Commitment," *Philosophical Studies* XI (1960): 3–7, with the kind permission of the editors and of the University of Minnesota Press

there are two considerations," Church says, "which combine to suggest that ontological commitment should be associated specifically with the existential quantifier rather than with bound variables generally." If the rules of quantification of L are modified so as to allow an interpretation within the null domain, it seems that "ontological commitment will attach to an existential statement, but not to the negation of an existential statement or to a universal statement." Secondly, Church wishes a sentence such as '$(\mathrm{E}x)x > 10^{1000}$', where the range of the variables is understood to be the positive integers, to give ontological commitment not to all positive integers, as with Quine, but merely to positive integers greater than 10^{1000}. Thirdly, Church speaks of the ontological commitment of sentences as pertaining to their *assertion* rather than to the sentences themselves.

Church's emendation of Quine's criterion seems therefore to be the following.

(C) The assertion of '$(\mathrm{E}x)$--x--', where '--x--' is a formula containing 'x' as its only free variable, carries ontological commitment to entities x such that --x--.

Let us consider the three points involved in this emendation. First what is meant by an assertion? Are we to think of assertions in some vague Fregean sense or are we to think of them as explicitly relativized to a person and a time? Perhaps we may say that person X *asserts* a sentence a of L at time t if and only if he *utters* a (in some suitable sense) at t and simultaneously *accepts* a to a sufficiently high degree.[3] But however this may be, without a characterization of what is involved in assertion, Church's third requirement is not clear.

The question also arises as to how we are to construe in (C) the phrase 'entities x such that --x--'. The most natural and immediate way is perhaps to regard it as standing for the *class* of all entities x such that --x--. In a footnote, however, Church comments that "ontological commitment must be to a class-concept rather than a class. For example, ontological commitment to unicorns is evidently not the same as ontological commitment to purple cows, even if by chance the two classes are both empty and therefore identical." Thus, apparently, we are not to construe 'entities x such that --x--' in (C) as standing

[3] Cf. *Toward a Systematic Pragmatics*, p. 70.

for a class, but for a class-concept instead. A more accurate statement of what Church apparently intends by (C) is then perhaps the following.

(C′) The assertion of '(Ex)--x--' carries ontological commitment to the class-concept of the class $\hat{x}$--x--.

But this is surely very different from (C), and it is not clear why (C) is given as the criterion for ontological commitment rather than (C′), if this latter is intended. (C′) or some variant thereof might better be regarded as a criterion for what might be called *conceptual* or *intensional* commitment. We shall attempt to characterize several types of intensional commitment in the next section.

Church's first point is in effect that we should take account of the null domain. But this seems somewhat gratuitous. The L's of Chapter I have been formulated in such a way as to require the existence of at least one object in the fundamental domain. In all applications of logic this assumption is not only harmless but desirable. The null domain may in fact be regarded as a mere idle mathematical curiosity with little or no philosophical interest.

Church's second point is that the ontological commitment (in some sense) of a sentence such as '(Ex)$x > 10^{1000}$', where 'x' ranges over positive integers, should be not to all positive integers but to just the positive integers $> 10^{1000}$. It is surely reasonable to require this of *some* kind of commitment. But we need not therewith give up Quine's original criterion. Let us rather use Church's second point as a preliminary statement of another kind of commitment, which we may call *existential* commitment.

First we consider existential commitment for sentences of L. This we may define, in the semantical metalanguage for L of (I, C), as follows.

'a ExisCmtmt $\boldsymbol{x} \ni A$' abbreviates '(Eb)(Ec)(SentFuncOne b, c . a = (c *exisqu* b) . ($\boldsymbol{x}$)(($c \cap$ *invep* $\cap b$) Den $\boldsymbol{x} \ni A$))', where A is a formula of L containing $\boldsymbol{x}$ as its only free variable.

Here the abstract '$\boldsymbol{x} \ni A$' stands for some virtual class of objects and 'SentFuncOne b, c' expresses that b is a sentential function of the one variable c, i.e., is a formula containing c as its only free variable (if any). According to this definition then we may say that an existential sentence

(*c exisqu b*) of *L carries existential commitment to* a given virtual class provided the corresponding abstract ($c \cap invep \cap b$) denotes all and only the members of that virtual class.

More special types of existential commitment may be introduced by requiring in the definiens here that *a* be *analytic*. This we may call existential *analytic* commitment. Similarly we may define still further types using 'Synthc', 'Tr', or 'Thm' in place of 'Anlytc'. In this way we characterize existential *synthetic* commitment, existential *veridical* commitment, and existential *theoremic* commitment respectively.

Thus, e.g., '$(\mathrm{E}x)x > 10^{1000}$' is Tr and a Thm, in some suitable *L* for arithmetic, and hence carries existential veridical and theoremic commitment to the class of integers $> 10^{1000}$. But it is not analytic, and hence carries only existential synthetic, but not analytic, commitment to that class.

We have defined 'ExisCmtmt' here only for the *L*'s of Chapter I. But the analogous definitions (one for each type) can also be given for *T* in a suitable metalanguage for *T*.

E. INTENSIONAL COMMITMENT

Although we have seen that Church's "emendation" to Quine's criterion is not itself acceptable, it nonetheless contains suggestions of interest. Let us attempt now to characterize certain kinds of *intensional* commitment, suggested by Church's concern with commitment to class-concepts.

Here we have no class-concepts in Church's sense. In their place we have the various types of objective and Whiteheadian intensions. Corresponding to some of these we may introduce types of intensional commitment as follows.

Let the system concerned be *T* and the semantical metalanguage that for *T* based on designation.

First, we wish to define what it might mean to say that an existential sentence '(E*x*)--*x*--' of *T* carries intensional *analytic* commitment to a given Whiteheadian intension. For this we let

'*a* $\mathrm{IntAnlytcCmtmt}_{\mathrm{ClsCon}}\ \mu$' abbreviate '(E*b*)(E*c*)(SentFuncOne *b*, *c* . $\mathrm{Vdl}^1\ c$. $a = (c\ exisqu\ b)$. $\mu\ \mathrm{WhtdAnlytcInt}_{\mathrm{ClsCon}}\ (c \cap circ \cap b)$)'.

An expression *a carries intensional analytic commitment to a class* μ *of classes of classes* provided *a* is an existential sentence of the form (*c exisqu b*), where *c* is a variable for individuals and μ is the Whiteheadian analytic intension of the class constant ($c \cap circ \cap b$). And similarly for class constants of higher logical types.

In analogous fashion we characterize intensional *synthetic* commitment, intensional *veridical* commitment, and intensional *theoremic* commitment.

Let us suppose for the moment that 'purple cow' and 'unicorn' are distinct primitive class constants of *T*. Let these be abbreviated respectively by 'PC' and 'U'. The Whiteheadian analytic intension of 'PC' is then not the same as that of 'U', because, to give just one instance, PC is analytically a member of $\hat{M}M = \mathrm{PC}$ but U is not. Thus the sentences '(E*x*)PC*x*' and '(E*x*)U*x*' carry intensional analytic commitment to different classes of classes of classes. But clearly they carry existential commitment to the same class Λ^1.

Further types of intensional commitment emerge by taking into account objective intensions of Leonardian type. Some of these are perhaps of only minor interest, the objective analytic and synthetic ones no doubt being the most important.

F. PROPOSITIONS

Let us return now for a moment to the notion of *proposition*. We have characterized quasi-propositions above as virtual classes of L-equivalent sentences. Now that genuine intensions are available for individual and predicate constants, it is appropriate to ask whether there are such things as *genuine* propositions as a nonlinguistic kind of entity. Let us reflect upon this a little, relative to the system *T*.

Let us consider only sentences of *T* containing no quantifiers, and first only atomic sentences. Let 'a' be a primitive InCon and 'R' a primitive ClsCon. Is there such a thing as a genuine proposition corresponding to the atomic sentence 'Ra'? A genuine proposition, it might reasonably be presumed, should depend entirely upon, i.e., be a function of, the intensions of some of the expressions constituting the corresponding sentence. Now that genuine intensions are available, genuine propositions ought therefore to be forthcoming in a fairly

straightforward way. Corresponding to 'R' we have several kinds of genuine objective and Whiteheadian intension, and similarly for 'a'. The atomic sentence 'Ra' involves just the two constants 'R' and 'a'. The corresponding proposition may thus perhaps be regarded as a couple of some kind, say an *ordered* couple, the order reflecting the order of occurrence of 'R' and 'a' in 'Ra'. (We need not require the components of the couple to be of the same logical type.) Actually there are thirty-two ordered couples to consider: that consisting of the $\mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}$ of 'R' with the $\mathrm{ObjAnlytcInt}_{\mathrm{InCon}}$ of 'a', that consisting of the $\mathrm{ObjAnlytcInt}_{\mathrm{ClsCon}}$ of 'R' with the $\mathrm{ObjVerInt}_{\mathrm{InCon}}$ of 'a', and so on, including Whiteheadian intensions of ClsCon's. Each of these couples would have the crucial characteristic of a proposition corresponding to 'Ra', namely, that it is uniquely determined in some way by the intensions of 'R' and 'a'.

By suitable technical devices this kind of treatment may be extended to molecular sentences containing no quantifiers. Also it may be extended to molecular sentences containing quantifiers having only a restricted range, perhaps further. We should then have the task of studying these various kinds of propositions in detail and their precise relations with the quasi-propositions.

We shall not baptize this plethora of propositions with Christian names, nor need we inquire further into their role in semantics. We merely note them here as theoretical possibilities, leaving for subsequent inquiry their fuller characterization. In spite of nearly two thousand years of logical development, the exact study of propositions upon a clear foundation in syntax and semantics seems still in its infancy.

G. SYNTACTICAL, SEMANTICAL, AND PRAGMATICAL SYSTEMS

Throughout this book we have been concerned with the language-systems L and T as object-languages. These have been discussed at different levels, as it were. When we speak only of the syntax of L or T, we are in effect regarding L and T as *syntactical systems*. When we include also their semantics, we are regarding them as *semantical* systems. We have observed that the semantics of L or T is completely determined by the Rules of Denotation or Designation appropriate for that system, as formulated within a suitable metalanguage. We

have also noted that, if the notion of analytic truth for *L* or *T* is available and the metalanguage is of sufficient power, the intensional semantics of *L* or *T* is also forthcoming.

Now that a quantitative pragmatics for these systems is also provided, we have what we may call *pragmatical systems.* These, namely, are language-systems regarded as determined not just by syntactical and semantical rules and definitions, but by the pragmatical rules as well. Recent analytic philosophy has been much concerned with problems connected with the *pragmatics of systems*, with their acceptance or rejection as a whole, with the holistic character of a scientific theory, and so on. But no systematic pragmatics of systems has been developed or even so much as outlined or hinted at in the recent literature. The reason is perhaps not so much distrust of pragmatics itself as it is a loss of philosophical interest in language-systems in favor of natural languages. But this loss of interest in systems is tantamount to the rejection of logic itself. The systems are there, imbedded in language, whether we like them or no. However this may be, the foregoing material seems to provide a basis for developing an urgently needed pragmatics of systems.

In (II, C) and (III, A) above we have distinguished *General* and *Specific Rules* concerning Prfr and Eq. We noted that the General Rules lay down conditions holding for all users *X*, for all sentences *a*, *b*, and *c*, and at all times *t*. The Specific Rules concerning Prfr and Eq, on the other hand, stipulate conditions holding only for specific persons or sentences or times. Constants for human beings, sentences, and/or times must thus presumably occur in them. The Specific Rules are in effect observation sentences, or generalizations of such, made by the experimenter *E*.

The definitions concerning subjective intension and quasi-intension have all been relativized to a given virtual class *F* of sentences of especial interest to the experimenter *E* at the time. The larger and more varied *F* is, the better, of course. The case is not excluded where *F* consists of all the sentences of *L* or *T* of suitable form.

In characterizing pragmatical systems we must take into account both the General and Specific Rules concerning Prfr and Eq as well as the relativization to the virtual class *F*. At any given time the pragmatical description of the system is necessarily incomplete. It can take

account only of the times to date, of sentences in F, and of human beings or social groups under observation at the time. At every moment the description may change, just as it may for every new person whose preferences are observed, and for any enlargement of F. But this is what we should expect in the pragmatical description of a language, as in scientific observation and description generally.

A second kind of pragmatical system can be characterized within a pragmatics presupposing only a syntax. Here no semantical axioms or definitions are presupposed, but merely the syntactical description of the system together with the General and Specific Prfr- and Eq-Rules.

We must of course regard language-systems as instruments or tools for human use and recognize that this use itself is an immensely complex phenomenon which in turn needs an analysis and clarification. The foregoing provides the materials needed to characterize some aspects of such use in detail. More generally, it may be regarded as providing some of the key notions needed in the logical foundations of philosophical analysis.

H. THE MULTIDIMENSIONALITY OF MEANING

The word 'meaning' has been eschewed throughout as a technical term. Occasionally it has been used rather loosely in saying, e.g., that such and such an intension approximates to some extent a notion of meaning, and occasionally it has been used—rather incorrectly—as a synonym for 'intension' itself. Now that a full-fledged theory of intensions is available, however, we may explicitly contrast the two. Let us reflect then upon how 'meaning' might be introduced, within some semantical or pragmatical metalanguage, as a technical term.

Whatever meanings are, they have a rather complex structure. Most logical theories of meaning have oversimplified to the extreme in the sense of taking meanings—Frege's *Sinne*—as indivisible wholes. Such theories have failed to do justice, it would seem, to the many-faceted multidimensionality of meaning.

Given any individual or class constant, one factor or dimension of

its meaning is surely its analytic intension, in the Whiteheadian sense say. Perhaps even this is the most important factor, but it is surely one. Another factor no doubt is the Whiteheadian veridical intension, and still another the theoremic one. The Whiteheadian synthetic intension also should be listed. Meaning surely has a social dimension in some fashion, so that societal intensions, restricted perhaps to certain time intervals, must be taken into account. Here, no doubt, societal analytic, veridical, and also synthetic, theoremic, and contingent intensions of suitable degree and relative to suitable F must be included.

We have spoken here only of Whiteheadian intensions and not of the objective ones (of Leonardian or comprehensional type). When we speak of the meaning of a ClsCon, of a ClsClsCon, and so on, perhaps these comprehensional intensions should be brought in also. The Whiteheadian intensions have the richer membership, as we have already observed, and in a kind of way comprise the comprehensional ones. Thus the latter are perhaps not strictly needed. However, they may easily be included here if desired.

There seem to be, then, at least these nine factors in meaning, perhaps more. If we assume for the moment that there are just these nine, we may then introduce meanings technically as *ordered 9-ads* (enneads or nontuples) of these intensions of the same term. If we find in general n irreducible factors, we may introduce meanings as ordered n-ads, assuming of course that the pragmatical metalanguage is sufficient to handle such.

If we should wish to abstract from the pragmatical dimensions and concentrate on the purely semantical ones, we may perhaps regard meanings as ordered quadruples of the Whiteheadian analytic, synthetic, veridical, and theoremic intensions, or as ordered octuples (8-ads or octads) of these with objective analytic, synthetic, veridical, and theoremic ones. Such a purely semantical account of meaning would surely have uses, although for many purposes we should wish to take account of the pragmatical factors also. Of course we are speaking very roughly, indeed merely heuristically, with no attempt at exactitude.

Very little has been said here concerning the behavioral aspect of meaning, except in the discussion of the primitives 'Prfr' and 'Eq'. Human beings behave in various ways and their behavior is intimately

related with their use of language. We might thus wish to bring in behavior as an explicit factor, but just how is not too clear.[4]

Note that the theoremic intensions have been expressly included as a dimension of meaning. This seems justified, to some extent at least, in virtue of *TC6* (V) and allied theorems. Also there is an important connection here with Carnap's notion of a *meaning postulate*.[5] In general, the nonlogical axioms of L or T are formulae laying down various properties of the nonlogical primitives. Some of these may be statements of fact, but some of them may merely stipulate part of the meaning of the primitive—in accord with our decision to use the primitive in such and such a way (as Carnap would put it). Thus suppose 'W' ('warmer than') is a primitive of L standing for a dyadic relation. Clearly then

$$\text{`}(x){\sim}\mathrm{W}xx\text{'}$$

and

$$\text{`}(x)(y)(z)((\mathrm{W}xy\,.\,\mathrm{W}yz) \supset \mathrm{W}xz)\text{'},$$

if taken as axioms, would serve merely to postulate some features of the meaning of 'W'. The Whiteheadian theoremic intension of 'W' then has as members both

$$\hat{M}(x){\sim}Mxx \quad \text{and} \quad \hat{M}(x)(y)(z)((Mxy\,.\,Myz) \supset Mxz).$$

And in general if '--W--' is a theorem containing no free variables, the Whiteheadian theoremic intension of 'W' has $\hat{M}$--M-- as a member.

We may say here that the theorem '--W--' *determines* the class of classes $\hat{M}$--M-- as a member of the Whiteheadian theoremic intension of 'W'. Each meaning postulate determines a member as do also the theorems (containing no free variables) which are logical consequences of the meaning postulates. (If we wish to single out just those classes of classes determined by the meaning postulates concerning 'W', thus eliminating from consideration analytic sentences, we could form the class difference (μ—ν) here, where μ is the Whiteheadian theoremic and ν the analytic intension of 'W'.) If further nonlogical primitives

[4] Some exploratory steps in this direction are taken in the author's "Performance, Purpose, and Permission," forthcoming in *Philosophy of Science*.

[5] See "Meaning Postulates," in *Meaning and Necessity*, pp. 222–29.

are introduced together with further meaning postulates upon them, including some which would serve to interrelate 'W' with the new primitives, the situation is of course somewhat more complicated.

Having meanings now available in an exact fashion, we have further kinds of *propositions* to consider, namely, as ordered couples of meanings for atomic sentences of *L* containing one-place primitive predicates, as ordered triples of meanings for atomic sentences containing two-place primitive predicates, and so on. Further types of *commitment* arise also, commitment now to meanings rather than to intensions.

We need not reflect upon these notions any longer. But enough has been said surely to suggest that the theory of meaning, in the exact technical sense here considered, should be further developed and indeed is an "untilled field crying for workers."

I. INTENSIONS, MEANING, AND ORDINARY LANGUAGE

In discussing the semantics of ordinary language we should tread lightly, in view of the difficulties noted in (IV, I). Nonetheless it may be of interest to observe whether, and if so how, the theory of intensions and meaning developed or suggested above may be applicable to an ordinary language *NL*.

One difficulty to be faced is whether *NL* may be regarded as containing—and if so, in what sense—a theory of types. If it does contain such, we could presumably proceed as above. But that *NL* contains a theory of types seems highly dubious, or at any event problematic. And if it does, the precise sense in which it does surely needs to be characterized.

The theory of genuine intensions requires fundamental use of class- or set-theoretic notions. But it does not depend specifically upon distinctions of type, except as a means of avoiding contradiction. Some philosophical logicians regard some form of axiomatic set theory as more "natural," as a foundation for mathematics at least, than the theory based on types. The question then arises as to whether any form of set theory may be used in place of type theory as a foundation for the theory of intensions. We may then ask whether or in what sense the resulting theory may be said to be contained in *NL*. Perhaps *NL* can be "regimented" so as to contain some form of set theory,

resulting in a kind of "semi-ordinary" language.[6] Could the foregoing theory of genuine intensions be developed for the declarative sentences of such a regimented semi-ordinary language? We need not even attempt here to answer this difficult question, which anyhow would require results to be found perhaps in empirical linguistics.

It would be grossly to misunderstand, however, if it were thought that language-systems and the theory of quasi-intensions, intensions, and meaning are of interest only as approximations to, or as preparatory studies for, some theory or other about natural language. Indeed the same can be said of logic in general. On the contrary, natural language is not of especial interest from the point of view of logical theory; it is too amorphous, contains too many exceptions, is logically too imperfect, its aim and functioning are so different, and so on. If logic, including the theory of intensions and meaning, should prove to be of help in some wise to the student of natural language, this is all to the good. But this is not essential and is not the primary aim.

We have by no means exhausted the development of the theory of the various kinds of intension, of how they enter into meaning, of their interrelations, of their use in science, methodology, and various areas of philosophy, and of their connection with notions of ordinary language. But at least we have opened up the subject for subsequent investigation, and have shown its intimate connection—not heretofore brought to light, it would seem—with modern syntax, denotational semantics, and decision theory.

The reader may object to the complexity and technicality met in or presupposed by many of the foregoing definitions and explanations. But simpler alternatives are not known to be possible. In fact, the foregoing, whatever its defects, probably provides the logically simplest theory of intensions, both subjective and objective, which has yet been put forward. Readers who object to the technical details could perhaps show how to avoid them without loss of rigor. Readers who object to using logical systems at all are reminded that such systems may be viewed as mere fragments of natural language, and hence that any adequate theory of intensions and meaning for natural language

[6] See W. V. Quine, *Word and Object*, *passim*. Cf. also the author's "Existential Quantification and the 'Regimentation' of Ordinary Language."

should contain *ipso facto* an adequate theory for these fragments. Let us at least shore these fragments against our ruin. The logical complexities and logical systems are not dreamed up artificially; they are simply there, embedded in natural language, as it were, waiting to be brought to light and suitably characterized. Casual methods of analysis do not suffice, as Carnap and Church and others have repeatedly emphasized.

Over the portals of the entrance to contemporary philosophy is writ: Enter here fully equipped with the tools of the new logic. We only beg the reader, to paraphrase Russell in a famous passage, not to make up his mind against the kind of analyses given here—as he might be tempted to do because of the plethora of detail—until he has attempted to formulate a logic of meaning and of intensions (both subjective and objective) of his own. He will then find that the details are forced upon him willy-nilly if he works at the necessary depth and with the requisite logical care.

AUTHOR INDEX

INDEX OF SYMBOLS AND ABBREVIATIONS

SUBJECT INDEX

Date Due

Due	Returned	Due	Returned
LD May 3/64			
MAY 22 '64 ML(F)	[illegible] 6 [illegible]		

KEEP CARD IN POCKET

IT IS IMPORTANT THAT
CARD BE KEPT IN POCKET